REGULARITY AND RIVIÈRE'S $GL(m)$-GAUGE CONSTRUCTION FOR ELLIPTIC SYSTEMS WITH ANTISYMMETRIC POTENTIALS IN ARBITRARY DIMENSIONS.

CAROLIN BAYER

Abstract. Let $1 \leq q \leq 2$ and denote by $2 \leq q'$ its corresponding conjugate exponent. We prove the continuity of solutions $u \in W^{1,(\frac{n}{n-1},q')}(B^n,\mathbb{R}^m)$ to the critical elliptic system $-\Delta u = \Omega \cdot \nabla u$ in dimension $n \geq 3$, where the potential $\Omega \in L^{(n,q)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ is antisymmetric. First, we construct $P \in W^{1,(n,q)}(B^n, \mathrm{SO}(m))$ such that the PDE can be rewritten as $-\operatorname{div}(P^{-1}du) = *d\xi \cdot P^{-1}du$, which is nearly a Jacobian structure up to the rotation P. Second, we provide a Rivière's GL(m)-Gauge in order to establish a "full" (A, B)-conservation law, i.e. $-\operatorname{div}(Adu) = d^*B \cdot du$. We show that the assumption on $\Omega \in L^{(n,q)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ for $q \leq 2$ is optimal.

Contents

1. Introduction

Motivation of the problem. The regularity question — whether weak solutions of the harmonic map equation are smooth in the conformally invariant case for $n = 2$— was settled by Hélein [8], who used moving frames to handle the nonlinearity of the right-hand side. For maps into symmetric targets, this structure reduces to a Jacobian expression,

Department of Mathematics, ETH Zentrum, CH-8093 Zürich, Switzerland.
The author is funded by the Swiss National Fund, Project SNF 200020_219429.

and continuity follows from the Wente inequality [25]. More precisely, let $B^2 \subset \mathbb{R}^2$ and let $a, b \in W^{1,2}(B^2)$. Consider the system

$$\begin{cases} \Delta\varphi = \nabla a \cdot \nabla^{\perp} b & \text{on } B^2, \\ \varphi = 0 & \text{on } \partial B^2. \end{cases} \tag{1.1}$$

Wente discovered improved regularity properties for the solution φ of (1.1) given by

$$\|\varphi\|_{L^\infty(B^2)} + \|\nabla\varphi\|_{L^2(B^2)} \leq C\|\nabla a\|_{L^2(B^2)}\|\nabla b\|_{L^2(B^2)}. \tag{1.2}$$

A generalization of these better integrability estimates goes back to the compensated compactness theory of Coifman, Lions, Meyer, and Semmes [5] for functions that lie in Hardy space $\mathcal{H}^1$ rather than merely L^1. They further showed that solutions of (1.1) satisfy

$$\|\nabla\varphi\|_{L^{2,1}(B^2)} + \|\nabla^2\varphi\|_{L^1(B^2)} \leq C\|\nabla a\|_{L^2(B^2)}\|\nabla b\|_{L^2(B^2)}. \tag{1.3}$$

Bethuel showed a similar version [4], namely

$$\|\nabla\varphi\|_{L^2(B^2)} \leq C\|\nabla a\|_{L^{2,\infty}(B^2)}\|\nabla b\|_{L^2(B^2)}. \tag{1.4}$$

The full geometric reach of this cancellation structure was discovered by Rivière [18] for $n = 2$, who proved that any conformally invariant elliptic system of the form $-\Delta u = \Omega \cdot \nabla u$ with antisymmetric potential $\Omega \in L^2$ can be rewritten as a so called "(A, B)-conservation law" / "Rivière's GL(m)-Gauge", yielding continuity without any assumption on the target.

The question naturally arises as to whether this structure provides improved regularity estimates, even in higher dimensions. Let $n \geq 3$. Rivière raised the question whether it remains true or not, that any solution $u \in W^{1,n}(B^n, \mathbb{R}^m)$ of

$$-\operatorname{div}\Big(|\nabla u|^{n-2}\nabla u\Big) = |\nabla u|^{n-2}\, \Omega \cdot \nabla u, \tag{1.5}$$

where $\Omega \in L^n(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ is also continuous. Recently, Martino and Schikorra and independently Schlagenhauf provided a negative answer to this question by giving a counterexample [13], [22]. It turns out that the natural extension of these results leads to the so-called Lorentz spaces, which were introduced in Section 2 and play the same crucial role in our result as they do in, for example, [12, 3].

Formulation of the problem. Let now $n \geq 3$ and $m \in \mathbb{N}$. Fix $1 \leq q \leq 2$ and $q' = \frac{q}{q-1} \geq 2$ its conjugate exponent. Let $u \in W^{1,(\frac{n}{n-1},q')}(B^n, \mathbb{R}^m)$ be a solution of the system

$$-\Delta u = \Omega \cdot \nabla u \quad \text{in } \mathcal{D}'(B^n), \tag{1.6}$$

where $\Omega \in L^{(n,q)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ is anti-symmetric. The right hand side of (1.6) becomes critical due to $L^{(\frac{n}{n-1},q')} \cdot L^{(n,q)} \hookrightarrow L^1$ and it corresponds to the endpoint case of the classical Calderón-Zygmund theory. In this setting, we establish better regularity, that is, the continuity of solutions of (1.6).

The assumption $\Omega \in L^{(n,q)}$ arises naturally in the proof (see (4.9)), as it is required to simultaneously obtain a bound on both – the potential $d\xi$ and the gauge dP. It is noteworthy that in the endpoint case $q = 2$, the assumption is the same as the one of Martino and Schikorra in their construction of a gauge for antisymmetric potentials for rotated n-harmonic systems [12]. At time of creation, the author was not aware of the publication of Guo, Xiang [7], which shows that the technique of the Rivière–Gauge construction also works in the case of $\Omega \in L^{(n,2)}$ and coincides with our endpoint case.

We present two approaches to establishing the continuity of $u \in W^{1,(\frac{n}{n-1},q')}$. In a first version given in Section 4, we employ duality estimates below the Lorentz exponent, in particular $L^{(\frac{n}{n-1},\infty)}$, which is only slightly larger than $L^{(\frac{n}{n-1},q')}$ and has already played a crucial role in the work of Bethuel [4] (see (1.4)) and [21]. This approach also illustrates that one can circumvent the need to establish the more challenging full (A, B)-conservation laws.
This first approach is inspired by Rivière and Struwe [20] for systems of the type (1.6), but under their assumption that $u \in W^{1,2}(B^n, \mathbb{R}^m)$ and $\Omega \in L^2(B^n, \mathfrak{so}(m) \otimes \wedge^1)$. In their setting, the system (1.6) becomes supercritical in dimensions $n \geq 3$ and they have to assume additional Morrey assumptions in order to establish an "almost Jacobian" structure. We now give the main theorem of the first approach.

Theorem 1.1. *Let $m \in \mathbb{N}$. Let $q \leq 2$ and denote by $q' = \frac{q}{q-1} \geq 2$ its conjugate exponent. For every $n \geq 3$, there exists $\varepsilon_0(n) > 0$ such that for every weak solution $u \in W^{1,(\frac{n}{n-1},q')}(B^n, \mathbb{R}^m)$ of the equation* (1.6) *and for every $\Omega \in L^{(n,q)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ satisfying the smallness assumption*

$$\|\Omega\|_{L^{(n,q)}(B^n)} < \varepsilon_0(n), \tag{A}$$

then there exist $\alpha \in (0, 1)$ such that $u \in C^{0,\alpha}_{\text{loc}}(B^n, \mathbb{R}^m)$.

In a second version given in Section 5, we show that it is indeed possible to construct a "full" (A, B)-conservation law/ Rivière's GL(m)-Gauge to obtain the same result. More precisely, instead of transforming the system by P^{-1}, as was done in the first approach, the PDE is now transformed by $A \in L^\infty \cap W^{1,(n,q)}(B^n, \mathrm{GL}(m))$ and we obtain

$$-\operatorname{div}(A\, du) = -dA \cdot du + A\Omega \cdot du = \underbrace{(-dA + A\Omega)}_{=:(i)} \cdot du. \tag{1.7}$$

Further, we show the existence of $B \in W^{1,(n,q)}(B^n, \mathbb{R}^{m\times m} \otimes \wedge^2)$ in order to write (i) as a purely co-exact expression, so that the right-hand side becomes a "full" Jacobian:

$$-\operatorname{div}(Adu) = \underbrace{d^*B}_{\in L^{(n,q)}} \cdot \underbrace{du}_{\in L^{(\frac{n}{n-1},q')}} . \tag{1.8}$$

The following theorem is the main result of this second approach and leads to optimal regularity.

Theorem 1.2. *Let $n \geq 3$ and $m \in \mathbb{N}$. Let $q \leq 2$ and denote by $q' = \frac{q}{q-1} \geq 2$ its conjugate exponent. There exists $\varepsilon_0(n) > 0$ such that for every weak solution $u \in W^{1,(\frac{n}{n-1},q')}(B^n, \mathbb{R}^m)$ of the equation (1.6) and for every $\Omega \in L^{(n,q)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ satisfying the smallness assumption*

$$\|\Omega\|_{L^{(n,q)}(B^n)} < \varepsilon_0(n), \tag{A}$$

we then have $u \in W^{2,p}_{\mathrm{loc}}(B^n, \mathbb{R}^m)$ for any $p < n$ and hence $C^{0,\alpha}_{\mathrm{loc}}(B^n, \mathbb{R}^m)$ for any $\alpha \in (0,1)$. Moreover, there exists a (A,B)–Gauge with $A \in L^\infty \cap W^{1,(n,q)}(B^n, \mathbb{R}^{m\times m})$ and $B \in W^{1,(n,q)}(B^n, \mathbb{R}^{m\times m} \otimes \wedge^2)$ satisfying

$$-d^*B := dA - A\Omega. \tag{1.9}$$

such that the weak solution $u \in W^{1,(\frac{n}{n-1},q')}(B^n, \mathbb{R}^m)$ of (1.6) satisfies the following conservation law

$$d(*Adu + (-1)^{n-1}(*B) \wedge du) = 0. \tag{1.10}$$

Remark 1.3. *We point out that Theorem 1.2 immediately implies Theorem 1.1.*

Novelty of this project. The novelty of this project lies in developing a more refined understanding of critical systems with antisymmetric potentials in higher dimensions. The paper relies heavily on the following type of estimate:

$$\|\varphi\|_{L^\infty(B^n)} + \|d\varphi\|_{L^{(n,1)}(B^n)} + \|\nabla^2\varphi\|_{L^{(\frac{n}{2},1)}(B^n)} \lesssim \|da\|_{L^{(n,q)}(B^n)}\|db\|_{L^{(n,q')}(B^n)}, \tag{1.11}$$

where $\varphi \in W^{1,(n,1)}(B^n, \mathbb{R}^m)$ is a solution of

$$\begin{cases} \Delta\varphi = *(da \wedge db) & \text{on } B^n, \\ \varphi = 0 & \text{on } \partial B^n, \end{cases} \tag{1.12}$$

and $a \in W^{1,(n,q)}(B^n, \mathbb{R}^m)$ and $b \in W^{1,(n,q')}(B^n, \wedge^{n-2})$. Estimate (1.11) can be interpreted as a "high-dimensional Wente" and should be compared with (1.3).

In Section 6 we show that the assumption on the integrability of $\Omega \in L^{(n,q)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ for $q \leq 2$ is indeed optimal. More precisely, one cannot assume a milder assumption on $\Omega \in L^{n,q}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ for some $q > 2$ in order to conclude continuity on u.

2. Preliminary

We provide a brief introduction to Lorentz spaces, which arise naturally as refinements of standard Lebesgue spaces. For further reading, we refer to [6, p.48].

Let $\Omega \subset \mathbb{R}^n$ be an open set. For any measurable function $f : \Omega \to \mathbb{R}$, we define its decreasing rearrangement $f^* : [0, \infty) \to [0, \infty)$ as

$$f^*(t) := \inf\{\lambda \geq 0 : |\{x \in \Omega : |f(x)| > \lambda\}| \leq t\} \quad \text{for } t > 0, \tag{2.1}$$

and its average as

$$f^{**}(s) := \frac{1}{s}\int_0^s f^*(t)dt, \qquad \text{for } s > 0. \tag{2.2}$$

Given exponents $p \in [1, \infty)$ and $q \in [1, \infty]$, the Lorentz quasi-norm is defined as

$$[f]_{L^{(p,q)}(\Omega)} := \begin{cases} \left(\displaystyle\int_0^\infty t^{\frac{q}{p}} f^*(t)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \displaystyle\sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty. \end{cases} \tag{2.3}$$

For $p \in (1, \infty)$ and $q \in [1, \infty]$, replacing f^* with f^{**} in the definition (2.3) yields an equivalent norm to (2.3), denoted by $\|f\|_{L^{(p,q)}(\Omega)}$ (see [2, Thm. 4.6, Chap. 4]).

Definition 2.1. *Let $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be an open set. For $1 < p < \infty$, $1 \leq q \leq \infty$, we define the Soboloev-Lorentz space as*

$$W^{k,(p,q)}(\Omega) := \left\{ f \in L^{(p,q)}(\Omega) : \partial^\alpha f \in L^{p,q}(\Omega) \text{ for each } 0 \leq |\alpha| \leq k \right\}. \tag{2.4}$$

It holds $L^p(\Omega) = L^{(p,p)}(\Omega)$. If $|\Omega| < \infty$ and $1 \leq q < p < r \leq \infty$, we have [2, Prop.4.2, Chap. 4]

$$\underbrace{L^{(r,r)}}_{=L^r(\Omega)}(\Omega) \overset{r>p}{\subset} L^{(p,r)}(\Omega) \overset{r<p}{\subset} \underbrace{L^{(p,p)}(\Omega)}_{=L^p(\Omega)} \overset{p<q}{\subset} L^{(p,q)}(\Omega) \overset{p>q}{\subset} \underbrace{L^{(q,q)}(\Omega)}_{=L^q(\Omega)}. \tag{2.5}$$

Several classical functional inequalities can be generalised to Lorentz spaces.

Theorem 2.2 (Hölder's Inequality for Lorentz Spaces.)**.** *Assume $f \in L^{(p_1,q_1)}(\Omega)$ and $g \in L^{(p_2,q_2)}(\Omega)$ for $1 < p_1, p_2, p < \infty$ and let $1 \leq q_1, q_2, q \leq \infty$, that satisfy*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \qquad \text{and} \qquad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q}. \tag{2.6}$$

Then $fg \in L^{(p,q)}(\Omega)$ and satisfies

$$\|fg\|_{L^{(p,q)}} \leq C(p_1, p_2, q_1, q_2)\|f\|_{L^{(p_1,q_1)}}\|g\|_{L^{(p_2,q_2)}}. \tag{2.7}$$

This result extends to the case where $(p_1, q_1) = (\infty, \infty)$, reducing to the standard L^∞ bound.

Proof. See [15, Thm. 3.4] □

Theorem 2.3 (Young's Inequality for Convolutions.)**.** *Assume $f \in L^{(p_1,q_1)}(\mathbb{R}^n)$ and $g \in L^{(p_2,q_2)}(\mathbb{R}^n)$ for $1 < p_1, p_2, p < \infty$ and let $1 \leq q_1, q_2, q \leq \infty$, that satisfy*

$$\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}, \qquad \text{and} \qquad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q}. \tag{2.8}$$

Then $f * g \in L^{(p,q)}(\mathbb{R}^n)$ *and satisfies*

$$\|f * g\|_{L^{(p,q)}} \leq C\|f\|_{L^{(p_1,q_1)}}(p_1, p_2, q_1, q_2)\|g\|_{L^{(p_2,q_2)}}. \tag{2.9}$$

Proof. See [15, Thm. 2.6]. □

Corollary 2.4 (Improved Sobolev Embeddings)**.** *Let* $n \geq 2$, $1 \leq p < n$, *and* $1 \leq q \leq \infty$. *Define the Sobolev conjugate exponent* $p^* = \frac{np}{n-p}$. *Then the Sobolev-Lorentz space* $W^{1,(p,q)}(\mathbb{R}^n)$ *embeds continuously into the Lorentz space* $L^{(p^*,q)}(\mathbb{R}^n)$*:*

$$W^{1,(p,q)}(\mathbb{R}^n) \hookrightarrow L^{p^*,q}(\mathbb{R}^n). \tag{2.10}$$

In particular, for standard Sobolev spaces where $q = p$, *we have* $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*,p}(\mathbb{R}^n)$, *and for the endpoint* $p = 1$,

$$W^{1,1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1},1}(\mathbb{R}^n). \tag{2.11}$$

Proof. See [16, p. 294, Thm. 4.1], [15, Thm. 2.6]. The case $p = 1$ can be find in [17] □

Proposition 2.5. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with Lipschitz boundary. Let* $1 \leq p < n$ *and* $1 \leq q \leq \infty$, *and* $p^* = \frac{np}{n-p}$ *its conjugate Sobolev exponent. For every* $u \in W^{1,(p,q)}(\Omega)$, *we have*

$$\|u - \bar{u}_\Omega\|_{L^{p^*,q}(\Omega)} \leq C\|\nabla u\|_{L^{p,q}(\Omega)}, \tag{2.12}$$

where $\bar{u}_\Omega = \frac{1}{|\Omega|}\int_\Omega u(x)\,dx$ *and* $C = C(n, p, q, \Omega) > 0$ *is a constant independent of* u.

Proposition 2.6. *Let* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *with* $\nabla f \in L^{(n,1)}(\mathbb{R}^n)$. *Then* f *is equivalent to a continuous function and satisfies*

$$\|f - c\|_{L^\infty} \leq C\|\nabla f\|_{L^{(n,1)}}, \tag{2.13}$$

where $c = c(f)$ *a suitable constant.*

The corresponding proofs can be found in [2, 15, 23].

3. Construction of a Gauge $P \in W^{1,(n,q)}(B^n, \mathrm{SO}(m))$.

The following Lemma provides a suitable gauge that, due to the better integrability assumption of $\Omega \in L^{(n,q)}$ compared to the gauge in [18], also inherits better integrability. The proof strategy remains the same as in [18] and [20].

Lemma 3.1. *Let* $1 \leq q \leq 2$. *There exists* $\varepsilon_0 > 0$ *and* $C > 0$ *such that for any* $\Omega \in L^{(n,q)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ *satisfying*

$$\|\Omega\|_{L^{(n,q)}(B^n)} \leq C\varepsilon_0, \tag{3.1}$$

then there exists $P \in W^{1,(n,q)}(B^n, \mathrm{SO}(m))$ *and* $\xi \in W^{1,(n,q)}(B^n, \mathfrak{so}(m) \otimes \wedge^{n-2})$ *such that*

$$\begin{cases} *d\xi = P^{-1}dP + P^{-1}\Omega P & \text{in } B^n, \\ d * \xi = 0 & \text{in } B^n, \\ \xi = 0 & \text{on } \partial B^n. \end{cases} \tag{3.2}$$

Moreover, the following a–priori bound holds:

$$\|dP\|_{L^{(n,q)}(B^n)} + \|d\xi\|_{L^{(n,q)}(B^n)} \leq C\|\Omega\|_{L^{(n,q)}(B^n)} \leq C\varepsilon_0. \tag{3.3}$$

We follow the approach of Uhlenbeck in [24, Lemma 2.7, Lemma 2.8]. As a first step, we prove the Lemma for a slightly more regular space, see Lemma A.1. This makes the map $\lambda \mapsto P^{-1}\lambda P$ smooth. Using an approximation argument, we derive Lemma 3.1 as a consequence of Lemma A.1.

Proof of Lemma 3.1. We will show that Lemma 3.1 follows directly from Lemma A.1.
Let $\Omega \in L^{(n,q)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$. We assume that $\|\Omega\|_{L^{(n,q)}(B^n)} < \varepsilon_0$ for some sufficiently small $\varepsilon_0 > 0$ and fix ε_0 later.
Let $\frac{n}{2} < p < n$ as in Lemma A.1 and recall that then $W^{1,(p,1)} \hookrightarrow L^{(n,q)}$. Due to the density of $C_c^\infty(B^n)$ in $L^{(n,q)}$, there exists a sequence $(\Omega_k)_{k\in\mathbb{N}}$ in $W^{1,(p,1)}(B^n)$ such that

$$\Omega_k \to \Omega \quad \text{strongly in } L^{(n,q)}.$$

We can construct this sequence using standard mollifiers, setting $\Omega_k := \eta_k * \Omega$. By Young's convolution inequality, we have:

$$\|\Omega_k\|_{L^{(n,q)}} \leq \|\eta_k\|_{L^1}\|\Omega\|_{L^{(n,q)}} = \|\Omega\|_{L^{(n,q)}} < C\varepsilon_0.$$

By choosing $C\varepsilon_0 \leq \varepsilon(n,m)$, we ensure that $\|\Omega_k\|_{L^{(n,q)}} < \varepsilon(n,m)$ for all k, satisfying the smallness assumption required for Lemma A.1.
Applying Lemma A.1 to each Ω_k, there exist sequences $P_k \in W^{2,(p,1)}(B^n, \mathrm{SO}(m))$ and $\xi_k \in W^{2,(p,1)}(B^n, \mathfrak{so}(m) \otimes \wedge^{n-2})$ satisfying the gauge system (A.4), as well as the uniform bounds (A.5) and (A.6).
Because the bounds in (A.6) are uniform with respect to k, we can extract weakly converging subsequences (the index is again denoted by k) such that $P_k \rightharpoonup P$ and $\xi_k \rightharpoonup \xi$ weakly in $W^{2,(p,1)}$. Passing to the limit as $k \to \infty$, we conclude the proof of Lemma 3.1. □

4. Proof of Theorem 1.1 – an "almost" Jacobian structure.

Proof of Theorem 1.1. Since continuity is a local property, it suffices to prove continuity on the ball $B_{1-\tau_0}(0)$ for an arbitrary $\tau_0 \in (0,1)$. Since the $L^{(n,q)}$-norm is absolute continuous, we can find a uniform radius $r_0 > 0$ such that for any $r < r_0$ and any $x \in B_{1-\tau_0}(0)$, we have $B_r(x) \subset B^n$ and

$$\|\Omega\|_{L^{(n,q)}(B_r(x))} \leq C\varepsilon_0. \tag{4.1}$$

During the proof, it is possible to adopt ε_0 as small as is necessary. By Lemma 3.1, this yields a local gauge $P \in W^{1,(n,q)}(B_r(x), \mathrm{SO}(m))$ and $\xi \in W^{1,(n,q)}(B_r(x), \mathfrak{so}(m) \otimes \wedge^{n-2})$ for which (3.2) holds on $B_r(x)$.

Applying this gauge transformation to du, we deduce:

$$\begin{aligned} -d^*(P^{-1}du) &= -dP^{-1} \cdot du + P^{-1}\Omega du \\ &= (P^{-1}dP + P^{-1}\Omega P)P^{-1}du \\ &= *d\xi \cdot P^{-1}du, \end{aligned} \tag{4.2}$$

where we used the identity $d(P^{-1}P) = 0$, which yield $-dP^{-1}P = P^{-1}dP$. In analogy with [20], equation (4.2) defines an "almost" Jacobian structure.

In this approach, we apply a duality argument inspired by [19, Thm. 3.5] and [20, p. 459]. Let $\delta \in (0,1)$ and consider an arbitrary 1-form $\phi \in L^{(n,1)}(B_{\delta r}(x), \wedge^1)$ with $\|\phi\|_{L^{(n,1)}(B_{\delta r}(x))} \leq 1$. Further, define the extension by zero $\widetilde{\phi} := \phi\chi_{B_{\delta r}(x)}$, so that $\|\widetilde{\phi}\|_{L^{(n,1)}(B_r(x))} \leq 1$.

Consider the Hodge decomposition (see [11, Cor. 10.5.1] for further details)

$$\widetilde{\phi} = d\varphi + d^*\psi + \mathfrak{h} \qquad \text{in } B_r(x), \tag{4.3}$$

where $d\varphi \in dW_T^{1,(n,1)}(B_r(x))$, $d^*\psi \in d^*W_N^{1,(n,1)}(B_r(x), \wedge^2)$,[1] and $\mathfrak{h} \in C^\infty(B_r(x), \wedge^1)$ is a harmonic 1-form satisfying $d\mathfrak{h} = 0$, $d^*\mathfrak{h} = 0$. Moreover, it holds:

$$\|d\varphi\|_{L^{(n,1)}(B_r(x))} + \|d^*\psi\|_{L^{(n,1)}(B_r(x))} + \|\mathfrak{h}\|_{L^{(n,1)}(B_r(x))} \leq C\|\widetilde{\phi}\|_{L^{(n,1)}(B_r(x))} \leq C. \tag{4.4}$$

We note briefly that this extension is required to ensure interior estimates for the harmonic part of $\widetilde{\phi}$ at a later stage. In addition, it enables us to combine all individual components in the final step.

Morrey decay estimate for $\|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_{\delta r}(x))}$

Since $P \in \mathrm{SO}(m)$, the norm is preserved, i.e., $|P^{-1}\nabla u| = |\nabla u|$. Using the duality between $L^{(n,1)}$ and $L^{(\frac{n}{n-1},\infty)}$, we can write:

$$\begin{aligned} \|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_{\delta r}(x))} &= \|P^{-1}\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_{\delta r}(x))} \\ &= \sup_{\substack{\phi \in L^{(n,1)}(B_{\delta r}(x),\wedge^1) \\ \|\phi\|_{L^{(n,1)}(B_{\delta r}(x))} \leq 1}} \int_{B_{\delta r}(x)} P^{-1}du \cdot \phi \, dvol_n. \end{aligned} \tag{4.5}$$

[1]This notation was introduced in [10, p. 38].
$dW_T^{1,(r,s)}(B_r(x)) := \{d\varphi \mid \varphi \in W^{1,(r,s)}(B_r(x)),\ \varphi|_{\partial B_r(x)} = 0\}$,
$d^*W_N^{1,(r,s)}(B_r(x), \wedge^2) := \{d^*\psi \mid \psi \in W^{1,(r,s)}(B_r(x), \wedge^2),\ *\psi|_{\partial B_r(x)} = 0\}$.

Fix an arbitrary test form $\phi \in L^{(n,1)}(B_{\delta r}(x), \wedge^1)$ with $\|\phi\|_{L^{(n,1)}(B_{\delta r}(x))} \leq 1$ with the extension $\widetilde{\phi}$ on $B_r(x)$ as explained above. Substituting the Hodge decomposition $\widetilde{\phi} = d\varphi + d^*\psi + \mathfrak{h}$, we obtain:

$$\begin{aligned}\int_{B_{\delta r}(x)} P^{-1}du \cdot \phi \, dvol_n &= \int_{B_r(x)} P^{-1}du \cdot \widetilde{\phi} \, dvol_n \\ &= \int_{B_r(x)} P^{-1}du \cdot (d\varphi + d^*\psi + \mathfrak{h}) \, dvol_n.\end{aligned} \tag{4.6}$$

We will estimate each of these three terms separately. In the last step, we take the supremum over all such $\phi \in L^{(n,1)}(B_{\delta r}(x), \wedge^1)$ and we conclude by combining them.

Estimate containing $d\varphi$.

Define $\bar{u}_{B_r(x)} := \fint_{B_r(x)} u$. Using Stokes' Theorem and the fact that $\varphi|_{\partial B_r(x)} = 0$, the boundary term vanishes, yields:

$$\begin{aligned}\int_{B_r(x)} d\varphi \wedge *P^{-1}du &= \int_{B_r(x)} d(\varphi * P^{-1}du) - \int_{B_r(x)} \varphi d(*P^{-1}du) \\ &= \int_{B_r(x)} d\xi \wedge (\varphi P^{-1})du \\ &= (-1)^{n-1}\int_{B_r(x)} d\big(d\xi \varphi P^{-1}(u - \bar{u}_{B_r(x)})\big) - \int_{B_r(x)} d\xi \wedge d(\varphi P^{-1})(u - \bar{u}_{B_r(x)}) \\ &= -\int_{B_r(x)} d\xi \wedge d(\varphi P^{-1})(u - \bar{u}_{B_r(x)})\end{aligned} \tag{4.7}$$

This implies in particular

$$\begin{aligned}\left|\int_{B_r(x)} d\varphi \wedge *P^{-1}du\right| &\lesssim \|d\xi\|_{L^{(n,q)}(B_r(x))}\Big(\|d\varphi\|_{L^{(n,q')}} + \|\varphi\|_{L^\infty}\|dP\|_{L^{(n,q')}}\Big) \\ &\qquad \cdot \big\|u - \bar{u}_{B_r(x)}\big\|_{L^{(\frac{n}{n-2},\infty)}} \\ &\leq C'\varepsilon_0(1 + C\varepsilon_0)\|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))},\end{aligned} \tag{4.8}$$

where we used $d(d\xi) = 0$, $W^{1,(n,1)} \hookrightarrow W^{1,(n,q)} \hookrightarrow W^{1,(n,q')}$ due to

$$q \leq q' \quad \Leftrightarrow \quad \frac{1}{q} \geq \frac{1}{q'} = 1 - \frac{1}{q} \quad \Leftrightarrow \quad 1 \leq q \leq 2, \tag{4.9}$$

the embedding $W^{1,(n,1)} \hookrightarrow L^\infty$ (see Prop. 2.6), estimate (3.3) together with (4.1), and the Lorentz-Hölder exponent relations (see Cor. 2.4):

$$\frac{1}{n} + \frac{1}{n} + \frac{n-2}{n} = 1, \qquad \frac{1}{q} + \frac{1}{q'} + \frac{1}{\infty} = 1. \tag{4.10}$$

Estimate containing $d^*\psi$.
Analogously to the above, using Stokes' Theorem together with the boundary condition $*\psi|_{\partial B_r(x)} = 0$, we obtain:
(4.11)
$$\int_{B_r(x)} P^{-1} du \wedge d*\psi = \int_{B_r(x)} \Big(d(P^{-1}(u - \bar{u}_{B_r(x)})) - dP^{-1}(u - \bar{u}_{B_r(x)}) \Big) \wedge d*\psi$$
$$= -\int_{B_r(x)} d\Big(d(P^{-1}(u - \bar{u}_{B_r(x)})) \wedge *\psi \Big) - \int_{B_r(x)} dP^{-1}(u - \bar{u}_{B_r(x)}) \wedge d*\psi$$
$$= -\int_{B_r(x)} dP^{-1}(u - \bar{u}_{B_r(x)}) \wedge d*\psi.$$
This implies in particular
(4.12)
$$\left| \int_{B_r(x)} P^{-1} du \wedge d*\psi \right| \lesssim \|dP\|_{L^{(n,q)}(B_r(x))} \|u - \bar{u}_{B_r(x)}\|_{L^{(\frac{n}{n-2},\infty)}(B_r(x))} \|d^*\psi\|_{L^{(n,q')}(B_r(x))}$$
$$\overset{(2.12)}{\leq} C'' \varepsilon_0 \|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))},$$
where we used the embedding (see Theorem 2.2)
$$L^{(n,q)} \cdot L^{(n,q')} \hookrightarrow L^{(n/2,1)}. \tag{4.13}$$

Remark 4.1. *We briefly explain why the assumption $\Omega \in L^{(n,q)}$ is indeed necessary and optimal. In* (4.11)*, it would actually be enough to assume only $dP \in L^n$, because $d^*\psi \in L^{(n,\frac{n}{n-1})}$ could compensate for this weaker integrability (recall that $d^*\psi \in L^{(n,1)}$). However, the stronger assumption becomes essential in* (4.7)*. There, if we only had $d\xi \in L^n$, we would need $d(\varphi P) \in L^{(n,\frac{n}{n-1})}$, which requires $dP \in L^{(n,\frac{n}{n-1})}$–a condition that not be satisfied in general.*

Estimate containing $\mathfrak{h}$.
Since $P \in \mathrm{SO}(m)$, we have:
$$\left| \int_{B_r(x)} P^{-1} du \wedge *\mathfrak{h} \right| \lesssim \|du\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))} \|\mathfrak{h}\|_{L^{(n,1)}(B_r(x))}. \tag{4.14}$$
In order to apply interior estimates for harmonic forms, we exploit the fact that we have extended the test function ϕ to $\widetilde{\phi}$, and we claim that for $\delta \in (0, \frac{3}{4})$ we have:
$$\|\mathfrak{h}\|_{L^{(n,1)}(B_r(x))} \leq C\delta^{n-1}. \tag{4.15}$$

Proof of the Claim. By a duality argument, we have:

$$\|\mathfrak{h}\|_{L^{(n,1)}(B_r(x))} = \sup_{\substack{k\in L^{(\frac{n}{n-1},\infty)}(B_r(x))\\ \|k\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))}\leq 1}} \int_{B_r(x)} \mathfrak{h}\cdot k \, dvol_n. \tag{4.16}$$

Consider an arbitrary but fixed $k = df + d^*g + \mathfrak{H} \in L^{(\frac{n}{n-1},\infty)}(B_r(x))$ with $df \in dW_T^{1,(\frac{n}{n-1},\infty)}(B_r(x))$, $d^*g \in d^*W_N^{1,(\frac{n}{n-1},\infty)}(B_r(x),\wedge^2)$, and $\mathfrak{H} \in C^\infty(B_r(x),\wedge^1)$ and satisfy

$$\begin{aligned}
&\|df\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))} + \|d^*g\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x)} + \|\mathfrak{H}\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x)}\\
&\leq C\|k\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x)} \leq C.
\end{aligned}\tag{4.17}$$

Since $\widetilde{\phi}$ is supported in $B_{\delta r}(x)$ with $\|\widetilde{\phi}\|_{L^{(n,1)}(B_r(x))} \leq 1$, orthogonality yields:

$$\begin{aligned}
\int_{B_r(x)} \mathfrak{h}\cdot(df + d^*g + \mathfrak{H})\, dvol_n &= \int_{B_r(x)} \mathfrak{h}\cdot\mathfrak{H}\, dvol_n\\
&= \int_{B_r(x)} \widetilde{\phi}\cdot\mathfrak{H}\, dvol_n = \int_{B_{\delta r}(x)} \widetilde{\phi}\cdot\mathfrak{H}\, dvol_n\\
&\leq \|\widetilde{\phi}\|_{L^{(n,1)}(B_{\delta r}(x))}\|\mathfrak{H}\|_{L^{(\frac{n}{n-1},\infty)}(B_{\delta r}(x))}\\
&\leq \|\mathfrak{H}\|_{L^{(\frac{n}{n-1},\infty)}(B_{\delta r}(x))}.
\end{aligned}\tag{4.18}$$

Since $\mathfrak{H}$ is harmonic, the following function is monotone increasing for any $1 \leq p < \infty$ [2]

$$r \mapsto \frac{1}{r^n}\int_{B_r(x)} |\mathfrak{H}|^p. \tag{4.19}$$

Following the strategy in [19, p. 145], we choose $\delta \in (0,1)$ independent of r such that $\delta r < \frac{3}{4}r$ holds, i.e., $\delta \in (0,\frac{3}{4})$, which yields

$$\begin{aligned}
&\left(\frac{1}{\delta r}\right)^n \int_{B_{\delta r}(x)} |\mathfrak{H}|^p \leq \left(\frac{1}{3r/4}\right)^n \int_{B_{3r/4}(x)} |\mathfrak{H}|^p\\
\Leftrightarrow &\int_{B_{\delta r}(x)} |\mathfrak{H}|^p \leq \left(\frac{4\delta}{3}\right)^n \int_{B_{3r/4}(x)} |\mathfrak{H}|^p.
\end{aligned}\tag{4.20}$$

Because $\mathfrak{H}$ is harmonic, all local L^p norms are comparable by interior estimates and the mean-value formula. By choosing $1 < p < \frac{n}{n-1}$, we have $L^{(\frac{n}{n-1},\infty)}(B_r(x)) \hookrightarrow L^p(B_r(x))$ on

[2]This is a direct consequence of that fact that for $1 \leq p < \infty$, we have $\Delta|\mathfrak{H}|^p \geq 0$ and hence $|\mathfrak{H}|^p$ is subharmonic. For details see A.3.

$B_r(x)$. Using $\|\mathfrak{H}\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))} \leq \|k\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))} \leq 1$, we obtain:

(4.21)

$$\begin{aligned}
\|\mathfrak{H}\|_{L^{(\frac{n}{n-1},\infty)}(B_{\delta r}(x))} &\lesssim (\delta r)^{\frac{n(p-1)}{p}-1}\|\mathfrak{H}\|_{L^p(B_{\delta r}(x))} \\
&\overset{(4.20)}{\leq} C(n,p)\delta^{\frac{n}{p}}(\delta r)^{\frac{n(p-1)}{p}-1}\|\mathfrak{H}\|_{L^p(B_r(x))} \\
&\overset{p<\frac{n}{n-1}}{\leq} C(n,p)\delta^{\frac{n}{p}}(\delta r)^{\frac{n(p-1)}{p}-1}\|\mathfrak{H}\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))}\|1\|_{L^{(\frac{np}{n-p(n-1)},p)}(B_r(x))} \\
&\leq C(n,p)\delta^{\frac{n}{p}}(\delta r)^{\frac{n(p-1)}{p}-1} r^{\frac{n-(n-1)p}{p}}\|\mathfrak{H}\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))} \\
&= C(n,p)\delta^{n-1}\|\mathfrak{H}\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))}.
\end{aligned}$$

where we used Hölder inequality for the parameters (recall $p < \frac{n}{n-1}$)

$$\frac{1}{p} = \frac{n-1}{n} + \frac{1}{\frac{np}{n-p(n-1)}}, \qquad \frac{1}{p} = \frac{1}{\infty} + \frac{1}{p}. \tag{4.22}$$

Combining (4.16), (4.18), (4.21) and taking the supremum over $k \in L^{(\frac{n}{n-1},\infty)}(B_r(x))$ completes the proof of the claim. □

Combining (4.14) with (4.15) then yields

$$\left|\int_{B_r(x)} P^{-1}du \wedge *\mathfrak{h}\right| \leq C'''\delta^{n-1}\|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))}. \tag{4.23}$$

The final bound.

Combining the estimates (4.7), (4.11), and (4.23), taking the supremum over all $\phi \in L^{(n,1)}(B_{\delta r}(x), \wedge^1)$ with $\|\phi\|_{L^{(n,1)}(B_{\delta r}(x))} \leq 1$, and insert into (4.5), we end up with:

$$\|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_{\delta r}(x))} \leq \Big(\varepsilon_0(C' + C'') + \delta^{n-1}C'''\Big)\|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))}. \tag{4.24}$$

Fix an arbitrary exponent $\alpha \in (0, n-1)$. Since $n - 1 - \alpha > 0$, we can choose $\delta \in (0, 3/4)$ small enough such that

$$C'''\delta^{n-1} \leq \frac{1}{2}\delta^{\alpha} \quad \Leftrightarrow \quad \delta \leq \left(\frac{1}{2C'''}\right)^{\frac{1}{n-1-\alpha}}. \tag{4.25}$$

With $\delta \in (0, 3/4)$ now fixed, we choose $\varepsilon_0 > 0$ sufficiently small such that

$$\varepsilon_0(C' + C'') \leq \frac{1}{2}\delta^{\alpha}. \tag{4.26}$$

Combining (4.25) and (4.26) yields the following decay:

$$\|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_{\delta r}(x))} \leq \delta^{\alpha}\|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))}. \tag{4.27}$$

Iteration argument.
Let $0 < r < r_0$ be arbitrary. Then there exists $k \in \mathbb{N}_0$ such that

$$\delta^{k+1} r_0 \leq r < \delta^k r_0 \qquad \Leftrightarrow \qquad \delta^{k+1} \leq \frac{r}{r_0} < \delta^k. \tag{4.28}$$

Iterating (4.27) k times yield:

$$\begin{aligned}
\|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))} &\leq \|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_{\delta^k r_0}(x))} \leq \delta^{\alpha k} \|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_{r_0}(x))} \\
&\leq \delta^{\alpha(k+1)} \frac{1}{\delta^\alpha} \|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_{r_0}(x))} \\
&\overset{(4.28)}{\leq} \left(\frac{r}{r_0}\right)^\alpha \frac{1}{\delta^\alpha} \|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_{r_0}(x))} \\
&\overset{(4.25)}{=} \left(\frac{r}{r_0}\right)^\alpha (2C''')^{\frac{\alpha}{n-1-\alpha}} \|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_{r_0}(x))}.
\end{aligned} \tag{4.29}$$

Taking the supremum over $0 < r < r_0$ and $x \in B_{1-\tau_0}(0)$, we obtain for any $\alpha \in (0, n-1)$ a constant $C(n, \alpha, r_0) > 0$ such that

$$\sup_{\substack{x \in B_{1-\tau_0}(0) \\ 0 < r < r_0}} r^{-\alpha} \|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))} \leq C(n, \alpha, r_0) < \infty, \tag{4.30}$$

where $C(n, \alpha, r_0) := (2C''')^{\frac{\alpha}{n-1-\alpha}} r_0^{-\alpha} \|\nabla u\|_{L^{(\frac{n}{n-1},q')}(B^n)}$ and we used $L^{(\frac{n}{n-1},q')} \subset L^{(\frac{n}{n-1},\infty)}$.

Conclusion: Continuity via Morrey space techniques.
For any $x \in B_{1-\tau_0}(0)$ and $0 < r < r_0$, Hölder's inequality together with (4.30) yields:

$$\begin{aligned}
\|\nabla u\|_{L^1(B_r(x))} &\lesssim \|\nabla u\|_{L^{(\frac{n}{n-1},\infty)}(B_r(x))} \|1\|_{L^{(n,1)}(B_r(x))} \\
&\leq \tilde{C} r^\alpha \cdot r = \tilde{C} r^{\alpha+1},
\end{aligned} \tag{4.31}$$

where we used that $\|1\|_{L^{(n,1)}(B_r(x))} \lesssim r$.
Since (4.31) holds for any $\alpha < n-1$, choosing $n-2 < \alpha < n-1$ and defining $\mu := \alpha - n + 2 \in (0,1)$, the decay estimate can be rewritten as:

$$\|\nabla u\|_{L^1(B_r(x))} \leq \tilde{C} r^{n-1+\mu}. \tag{4.32}$$

By Morrey's Dirichlet Growth Theorem [14, Thm. 3.5.2], (4.32) implies Hölder continuity on the domain $B_{1-\tau_0}(0)$, i.e., $u \in C^{0,\mu}(B_{1-\tau_0}(0))$.
Since $\tau_0 \in (0,1)$ was arbitrary, we conclude that $u \in C^{0,\mu}_{\mathrm{loc}}(B^n)$, completing the proof of Theorem 1.1.

□

5. Proof of Theorem 1.2 – a "full" Conservation law.

The entire section follows the strategy of Rivière in [18, Thm. I.1, I.3, I.4].

Theorem 5.1. *There exists* $\varepsilon_0 > 0$ *and* $C_0 > 0$ *such that for any* $\Omega \in L^{(n,q)}(B^n, \mathfrak{so}(m)\otimes\wedge^1)$ *satisfying*

$$\|\Omega\|_{L^{(n,q)}} \leq C\varepsilon_0, \tag{5.1}$$

then there exists $A \in L^\infty \cap W^{1,(n,q)}(B^n, GL(m))$ *and* $B \in W^{1,(n,q)}(B^n, \mathbb{R}^{m\times m}\otimes\wedge^2)$ *satisfying*

$$\begin{cases} \|\operatorname{dist}(A, \mathrm{SO}(m))\|_{L^\infty(B^n)} \leq C_0\|\Omega\|_{L^{(n,q)}(B^n)}, \\ \\ \|d^*B\|_{L^{(n,q)}} \leq C_0\|\Omega\|_{L^{(n,q)}}, \\ \\ -d^*B = dA - A\Omega. \end{cases} \tag{5.2}$$

Proof. Since (5.1) is satisfied, we apply again Lemma 3.1 and we follow the existence of $P \in W^{1,(n,q)}(B^n, \mathrm{SO}(m))$, $\xi \in W^{1,(n,q)}(B^n, \mathfrak{so}(m)) \otimes \wedge^{n-2})$ which satisfies the system (3.2) as well as the a-priori bound:

$$\|d\xi\|_{L^{(n,q)}(B^n)} + \|dP\|_{L^{(n,q)}(B^n)} \leq C\|\Omega\|_{L^{(n,q)}(B^n)}. \tag{5.3}$$

We make the ansatz:

$$A := (\mathrm{Id}_m + \sigma)P^{-1} \qquad \Rightarrow dA = d\sigma P^{-1} + (I+\sigma)dP^{-1}, \tag{5.4}$$

for some $\sigma \in L^\infty \cap W^{1,(n,1)}(B^n, \mathbb{R}^{m\times m})$ with $\|\sigma\|_{L^\infty} < 1$. A direct consequence of the assumption is that A is invertible due to Neumann series. This assumption will be justified later, once the a-priori bounds (5.2) have been established.
We follow the strategy of Rivière in [18, Thm. I.4]. (Note that he constructs $(\mathrm{Id}_m + \sigma)$ by finding $\tilde{A}$ and σ takes the role of $\hat{A}$ there.)

We seek for A and B satisfying (5.2). Using ansatz (5.4) then yields

$$\begin{aligned} -d^*B = dA - A\Omega, \quad &\Leftrightarrow \quad -d^*B \overset{(5.4)}{=} d\sigma P^{-1} + (\mathrm{Id}_m+\sigma)dP^{-1} - (\mathrm{Id}_m+\sigma)P^{-1}\Omega \\ &\Leftrightarrow \quad -d^*BP = d\sigma - (\mathrm{Id}_m+\sigma)\underbrace{(-dP^{-1}P + P^{-1}\Omega P)}_{=*d\xi} \\ &\Leftrightarrow \quad -d^*BP = d\sigma - (\mathrm{Id}_m+\sigma) * d\xi. \end{aligned} \tag{5.5}$$

Applying the $d^*(\cdot)$-operator leads to

$$-\Delta\sigma + (-1)^{n-1} * (d\sigma \wedge d\xi) = d^*B \cdot dP. \tag{5.6}$$

Note that (5.5) is multiplying again by P^{-1} equivalent to

$$(I+\sigma) * d\xi P^{-1} = d^*B + d\sigma P^{-1}, \tag{5.7}$$

and hence by applying the $d(\cdot)$ operator (and recalling that $d(d\sigma P^{-1}) = -d\sigma \wedge dP^{-1} = -d(\sigma dP^{-1})$) this leads to the following system

$$d(*d\xi P^{-1}) + d(\sigma * d\xi P^{-1}) = \Delta B - d(\sigma dP^{-1}). \tag{5.8}$$

Combining (5.6) and (5.8), we then look for $\sigma \in W^{1,(n,q)}(B^n, \mathbb{R}^{m\times m})$ and $B \in W^{1,(n,q)}(B^n, \mathbb{R}^{m\times m}, \wedge^2)$ which solves

$$\begin{cases} -\Delta\sigma = (-1)^n * (d\sigma \wedge d\xi) + d^*B \cdot dP & \text{in } B^n, \\ \Delta B = d(*d\xi P^{-1}) + d(\sigma * d\xi P^{-1}) + d(\sigma dP^{-1}) & \text{in } B^n, \\ \partial_\nu \sigma = 0, \qquad B = 0 & \text{on } \partial B^n, \\ \displaystyle\int_{B^n} \sigma = \mathrm{Id}_m. \end{cases} \tag{5.9}$$

For the sake of simplicity, we omit the domain B^n. Standard elliptic estimates $W^{1,(\frac{n}{2},1)} \hookrightarrow L^{(n,1)}$ (see again Theorem 2.4), and Hölder's inequality gives:

$$\begin{aligned} \|d\sigma\|_{L^{(n,1)}} &\lesssim \|\Delta\sigma\|_{L^{(\frac{n}{2},1)}} \lesssim \|d\sigma\|_{L^{(n,q')}} \|d\xi\|_{L^{(n,q)}} + \|d^*B\|_{L^{(n,q')}} \|dP\|_{L^{(n,q)}} \\ &\overset{(5.3)}{\leq} C'\|\Omega\|_{L^{(n,q)}} \|d\sigma\|_{L^{(n,q')}} + C'\|\Omega\|_{L^{(n,q)}} \|d^*B\|_{L^{(n,q')}}, \end{aligned} \tag{5.10}$$

together with $W^{1,(n,1)} \hookrightarrow L^\infty$ and $L^{(n,q)} \subset L^{(n,q')}$ due to $2 \leq q \leq q'$, we have

$$\begin{aligned} \|d^*B\|_{L^{(n,q')}} &\lesssim \|d^*B\|_{L^{(n,q)}} \lesssim \|d\xi P^{-1}\|_{L^{(n,q)}} + \|\sigma * d\xi P^{-1}\|_{L^{(n,q)}} + \|\sigma dP^{-1}\|_{L^{(n,q)}} \\ &\lesssim \|d\xi\|_{L^{(n,q)}} \|P^{-1}\|_{L^\infty} + \|\sigma\|_{L^\infty} \|d\xi\|_{L^{(n,q)}} \|P^{-1}\|_{L^\infty} + \|\sigma\|_{L^\infty} \|dP^{-1}\|_{L^{(n,q)}} \\ &\leq C''\|\Omega\|_{L^{(n,q)}} + C''\|d\sigma\|_{L^{(n,1)}} \|\Omega\|_{L^{(n,q)}}. \end{aligned} \tag{5.11}$$

Choosing ε_0 small enough, by a fix-point argument we obtain the existence of B and σ, and hence also A, such that

$$\|d\sigma\|_{L^{(n,1)}} + \|\sigma\|_{L^\infty} + \|d^*B\|_{L^{(n,q)}} \leq C_0 \|\Omega\|_{L^{(n,q)}}. \tag{5.12}$$

Using (5.4), $\sigma = AP - \mathrm{Id}_m$, estimate (5.12) then leads to

$$\|AP - \mathrm{Id}_m\|_{L^\infty(B^n)} + \|d^*B\|_{L^{(n,q)}(B^n)} \leq C_0 \|\Omega\|_{L^{(n,q)}(B^n)}. \tag{5.13}$$

Again using the ansatz $A = (\mathrm{Id}_m + \sigma)P^{-1}$, estimate (5.12) further leads to

$$\begin{aligned} \|dA\|_{L^{(n,q)}} &\lesssim \|d\sigma P^{-1}\|_{L^{(n,q)}} + (1 + \|\sigma\|_{L^\infty}) \|dP^{-1}\|_{L^{(n,2)}} \\ &\overset{1\leq q}{\lesssim} \|d\sigma\|_{L^{(n,1)}} + C'\|\Omega\|_{L^{(n,q)}} + C''\|\Omega\|^2_{L^{(n,q)}} \\ &\lesssim C\|\Omega\|_{L^{(n,q)}} \end{aligned} \tag{5.14}$$

for a small enough $\varepsilon_0 > 0$. It remains to discuss the bound for dA^{-1}. We have $AA^{-1} = \mathrm{Id}_m$ and hence

$$dAA^{-1} + AdA^{-1} = 0 \quad \Rightarrow \quad dA^{-1} = -A^{-1}dAA^{-1}. \tag{5.15}$$

Equation (5.15) leads to

$$\|dA^{-1}\|_{L^{(n,q)}} \lesssim \|A^{-1}\|^2_{L^\infty}\|dA\|_{L^{(n,q)}}. \tag{5.16}$$

If we show that A^{-1} is bounded in L^∞, we can conclude by using (5.13). Since the Neumann series is well-defined by assumption, we have:

$$A^{-1} = ((\mathrm{Id}_m + \sigma)P^{-1})^{-1} = P\sum_{k=0}^{\infty}(-\sigma)^k. \tag{5.17}$$

Choosing $\varepsilon_0 < \frac{1}{2C_0}$, (5.17) is absolute-convergent and we get

$$\sum_{k=0}^{\infty}\|\sigma\|^k_{L^\infty} = \frac{1}{1-\|\sigma\|_{L^\infty}} \leq \frac{1}{1-C_0\varepsilon_0} \overset{\text{choice of } \varepsilon_0}{\leq} 2 < \infty, \tag{5.18}$$

It remains to show that the equality $-d^*B = dA - A\Omega$ holds.

Since B is a solution of (5.9), $(\mathrm{Id}_m + \sigma) * d\xi P^{-1} - d^*B - d\sigma P^{-1}$ is exact, and hence there exists a $C \in W^{1,(n,q)}(B^n)$ such that

$$\begin{aligned} &dC = (\mathrm{Id}_m + \sigma) * d\xi P^{-1} - d^*B - d\sigma P^{-1}, \\ \Leftrightarrow \quad &dCP = (\mathrm{Id}_m + \sigma) * d\xi - d^*BP - d\sigma. \end{aligned} \tag{5.19}$$

Applying the $d^*(\cdot)$-operator on (5.19), one gets

$$\Delta CP + dC \cdot dP = *(d\sigma \wedge d\xi) - d^*B \cdot dP - \Delta\sigma \overset{(5.9)}{=} 0. \tag{5.20}$$

We want to show that $C = 0$, from which (5.9) follows immediately.
Applying P^{-1} on both sides of (5.20) and using $dPP^{-1} = -PdP^{-1}$, C solves

$$\begin{cases} \Delta C = dC \cdot PdP^{-1} & \text{in } B^n, \\ C = 0 & \text{on } \partial B^n. \end{cases} \tag{5.21}$$

Using once more the Calderón-Zygmund estimates, we get

$$\begin{aligned} \|dC\|_{L^{(n,q)}} &\overset{1\leq q}{\lesssim} \|dC\|_{L^{(n,1)}} \lesssim \|\Delta C\|_{L^{(\frac{n}{2},1)}} \lesssim \|dC\|_{L^{(n,q)}}\|P\|_{L^\infty}\|dP\|_{L^{(n,q')}} \\ &\lesssim \varepsilon_0\|dC\|_{L^{(n,q)}}, \end{aligned} \tag{5.22}$$

and hence $\nabla C \equiv 0$. Due to the zero boundary conditions in (5.21), we conclude $C = 0$. This finishes the proof of Theorem 5.1.

□

We now give the continuity proof of solutions $u \in W^{1,(\frac{n}{n-1},q')}(B^n)$ to system (1.6).

Proof of Theorem 1.2. We argue analogously as in the proof of Section 4.
Since continuity is a local property, it suffices to prove continuity on the ball $B_{1-\tau_0}(0)$ for an arbitrary $\tau_0 \in (0,1)$. Since the $L^{(n,q)}$-norm is absolutely continuous, we can find a uniform radius $R_0 > 0$ such that for any $R < R_0$ and any $x \in B_{1-\tau_0}(0)$, we have $B_R(x) \subset B^n$ and $\Omega \in L^{(n,q)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ satisfies the smallness threshold given in Theorem 5.1:

$$\|\Omega\|_{L^{(n,q)}(B_R(x))} \leq C\varepsilon_0. \tag{5.23}$$

During the proof, it is possible to adopt ε_0 as small as is necessary. Theorem 5.1 yields the existence of $A \in L^\infty \cap W^{1,(n,q)}(B_R(x), \mathrm{GL}(m))$ and $B \in W^{1,(n,q)}(B_R(x), \mathfrak{so}(m) \otimes \wedge^2)$ such that

$$-d^*B = dA - A\Omega \qquad \text{in } B_R(x). \tag{5.24}$$

The conservation law claimed in (1.10) then follows by a direct computation:

$$\begin{aligned} d(*Adu) &= dA \wedge *du + A * \Delta u = dA \wedge *du - A\Omega \wedge *du \\ &\overset{(1.9)}{=} -d^*B \wedge *du = -(-1)^{n+1} * d * B \wedge *du \\ &= (-1)^n * d * B \wedge *du = -(-1)^{n-1} d * B \wedge du \\ &= -(-1)^{n-1} d((*B) \wedge du). \end{aligned} \tag{5.25}$$

Next, by a Hodge decomposition there exist $d\alpha \in dW_T^{1,(\frac{n}{n-1},2)}(B_R(x))$, $d^*\beta \in d^*W_N^{1,(\frac{n}{n-1},2)}(B_R(x), \wedge^{n-2})$, and $\mathfrak{h} \in C^\infty(B_R(x), \wedge^1)$ such that

$$\begin{cases} Adu = d\alpha + *d\beta + \mathfrak{h} & \text{in } B_R(x), \\ \alpha = *\beta = 0 & \text{on } \partial B_R(x), \\ \mathfrak{h} = Adu & \text{on } \partial B_R(x). \end{cases} \tag{5.26}$$

Using (5.25), α and β then solve:

$$\begin{cases} -\Delta\alpha = d^*(Adu) = d^*B \cdot du & \text{in } B_R(x), \\ \alpha = 0 & \text{on } \partial B_R(x), \end{cases} \tag{5.27}$$

and

$$\begin{cases} \Delta\beta = *d(Adu) = *(dA \wedge du) & \text{in } B_R(x), \\ *\beta = 0 & \text{on } \partial B_R(x). \end{cases} \tag{5.28}$$

Using again the endpoint interpolation of Riesz potentials together with $L^{(\frac{n}{n-1},1)} \hookrightarrow L^{(\frac{n}{n-1},q')}$ ($q' \geq 2$), we get the existence of a constant $C > 0$ such that

$$\begin{aligned}
\|d\alpha\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} &\lesssim \|d^*B\|_{L^{(n,q)}(B_R(x))}\|du\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} \\
&\leq C\varepsilon_0\|du\|_{L^{(\frac{n}{n-1},q')}(B_R(x))}, \\
\|d^*\beta\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} &\lesssim \|dA\|_{L^{(n,q)}(B_R(x))}\|du\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} \\
&\leq C\varepsilon_0\|du\|_{L^{(\frac{n}{n-1},q')}(B_R(x))}.
\end{aligned} \tag{5.29}$$

Morrey space decay estimate.

Let $\delta \in (0, \frac{3}{4})$. We follow the strategy of the proof in Section 4. Using the fact that $A \in \mathrm{GL}(m)$, (5.26) can be rewritten as $du = A^{-1}d\alpha + A^{-1} * d\beta + A^{-1}\mathfrak{h}$, we get:

$$\begin{aligned}
&\|\nabla u\|_{L^{(\frac{n}{n-1},q')}(B_{\delta R}(x))} \\
&\lesssim \|A^{-1}\|_{L^\infty(B_{\delta R})}\Big(\|d\alpha\|_{L^{(\frac{n}{n-1},q')}(B_{\delta R}(x))} + \|d^*\beta\|_{L^{(\frac{n}{n-1},q')}(B_{\delta R}(x))} + \|\mathfrak{h}\|_{L^{(\frac{n}{n-1},q')}(B_{\delta R}(x))}\Big) \\
&\overset{(\star)}{\lesssim} \|d\alpha\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} + \|d^*\beta\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} + \delta^{n-1}\|\mathfrak{h}\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} \\
&\overset{(5.26)}{\lesssim} \|d\alpha\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} + \|d^*\beta\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} \\
&\qquad + \delta^{n-1}\Big(\|Adu\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} + \|d\alpha\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} + \|d^*\beta\|_{L^{(\frac{n}{n-1},q')}(B_R(x))}\Big) \\
&\lesssim \Bigg(\varepsilon_0 + \delta^{n-1} + \|A\|_{L^\infty}\delta^{n-1}\Bigg)\|\nabla u\|_{L^{(\frac{n}{n-1},q')}(B_R(x))} \\
&\leq C'\Bigg(\delta^{n-1} + \varepsilon_0\Bigg)\|\nabla u\|_{L^{(\frac{n}{n-1},q')}(B_R(x))}.
\end{aligned} \tag{5.30}$$

The estimate in $(\star)$ needs to be discussed. Again, we use the monotonicity properties of harmonic functions/forms and refer the reader to the Proof in Section 4, specifically equation (4.20). In that argument, one may choose $p = \frac{n}{n-1}$, since the exponent does not need to be modified again on a larger ball and no further interior estimates are required. Note that due to the construction of A in Theorem 5.1, A^{-1} is bounded for the choice of $\varepsilon_0 < \frac{1}{2C_0}$ (see (5.18)):

$$\|A^{-1}\|_{L^\infty} = \|P(\mathrm{Id}_m + \sigma)^{-1}\|_{L^\infty} \lesssim \frac{1}{1-\varepsilon_0} \leq 2. \tag{5.31}$$

Fix an arbitrary exponent $\alpha \in (0, n-1)$. Since $n-1-\alpha > 0$, we can choose $\delta \in (0, 3/4)$ small enough such that

$$C'\delta^{n-1} \leq \frac{1}{2}\delta^{\alpha} \quad \Leftrightarrow \quad \delta \leq \left(\frac{1}{2C'}\right)^{\frac{1}{n-1-\alpha}}. \tag{5.32}$$

With $\delta \in (0, 3/4)$ now fixed, we choose $\varepsilon_0 > 0$ sufficiently small such that

$$C'\varepsilon_0 \leq \frac{1}{2}\delta^{\alpha}. \tag{5.33}$$

Iteration argument.
Following the iteration argument in Section 4, for any $\alpha \in (0, n-1)$ and constant $C(n, \alpha, R_0) > 0$, we end up with:

$$\sup_{\substack{x \in B_{1-\tau_0}(0) \\ 0<R<R_0}} R^{-\alpha} \|\nabla u\|_{L^{(\frac{n}{n-1}, q')}(B_R(x))} \leq C(n, \alpha, R_0) < \infty, \tag{5.34}$$

where $C(n, \alpha, R_0) \coloneqq (2C')^{\frac{\alpha}{n-1-\alpha}} R_0^{-\alpha} \|\nabla u\|_{L^{(\frac{n}{n-1}, q')}(B^n)}$ and we used $B_R(x) \subset B^n$.

Conclusion: Higher integrability of ∇u via Morrey-Adams techniques.
Let $x \in B_{1-\tau_0}(0)$, $R < R_0$, and $\alpha \in (0, n-1)$. Equation (5.34) leads to

$$\int_{B_R(x)} |\Delta u| \lesssim \|\Omega\|_{L^{(n,q)}(B_R(x))} \|\nabla u\|_{L^{(\frac{n}{n-1}, q')}(B_R(x))} \lesssim \|\Omega\|_{L^{(n,q)}(B_R(x))} R^{\alpha}. \tag{5.35}$$

Standard estimates by Adams [1, Prop. 3.2.(ii)] then yield $\nabla u \in L^s_{\mathrm{loc}}$ and hence $\nabla u \in L^{(\bar{s}, q')}_{\mathrm{loc}}$ for $s > \bar{s} > \frac{n}{n-1}$ due to $L^s \subset L^{(\bar{s}, q')}$. Indeed, by weak fractional integration, we conclude by Adams

$$\|\nabla u\|_{L^s(B_{R/2}(x))} \lesssim \|I_1(|\Delta u|)\|_{L^s(B_R(x))} < \infty. \tag{5.36}$$

We have thus improved the integrability of the gradient to $\nabla u \in L^{(\bar{s}, q')}_{\mathrm{loc}}(B_R(x))$ with $s > \bar{s} > \frac{n}{n-1}$. Consequently, the right-hand side of $-\Delta u = \Omega \cdot \nabla u$ becomes subcritical via Hölder's inequality on Lorentz spaces, $L^{(n,q)}_{\mathrm{loc}} \cdot L^{(\bar{s}, q')}_{\mathrm{loc}} \hookrightarrow L^r_{\mathrm{loc}}$ for some $r > 1$. By standard Calderón-Zygmund estimates and bootstrapping arguments, one obtains

$$\Delta u \in L^p_{\mathrm{loc}} \qquad \text{for every } n > p > \frac{n}{2}. \tag{5.37}$$

Note that the regularity of $\Omega \in L^{(n,q)}$ limits the regularity of Δu. Indeed, we can bootstrap $\nabla u \in L^s$ for any $s < \infty$ arbitrary large. However, due to the structure of the equation, we have that p is defined by

$$\frac{1}{s} + \frac{1}{n} = \frac{1}{p} \qquad \Leftrightarrow \qquad p = \frac{ns}{n+p}. \tag{5.38}$$

$p = p(s)$ is monotone increasing in s and hence bounded by $p = p(s) \to n$ for $s \to \infty$. Finally, by the Sobolev embedding $W^{2,p}_{\mathrm{loc}}(B^n) \hookrightarrow C^{0,\beta}_{\mathrm{loc}}(B^n)$ with $\frac{n}{2} < p < n$ and $\beta \coloneqq 2 - \frac{n}{p}$ we obtain Hölder continuity, which finishes the proof.

□

6. The Optimality of the Result.

Proposition 6.1. *Let $n \geq 3$. For every $\varepsilon_0 > 0$, there exist $u \in W^{1\frac{n}{n-1}}(B^n, \mathbb{R}^{n+1})$ and $\Omega \in L^n(B^n, \mathfrak{so}(n+1) \otimes \wedge^1)$ such that*

$$\|\Omega\|_{L^n(B^n)} < \varepsilon_0, \quad -\Delta u = \Omega \cdot \nabla u \quad \text{in } \mathcal{D}'(B^n), \tag{6.1}$$

but u is unbounded at the origin and hence u has no continuous representative.

Proof. We divide the construction and verification into several steps.

Step 1. Polar coordinates and elementary computations.
Write

$$r := |x| = \sqrt{\sum_{i=1}^{n} x_i^2}, \quad \theta := \frac{x}{|x|} \in S^{n-1}, \quad t := T + \log\frac{1}{r} = T - \log r, \tag{6.2}$$

where $T \geq 2$ will be chosen at the end. Differentiating t with respect to r gives $\frac{dt}{dr} = -\frac{1}{r}$, so $dt = -\frac{dr}{r}$. For $k \in \{1, \dots, n\}$, the functions $\theta_k = x_k/r$ satisfy the following.

Orthogonality $\langle d\theta_k, dr\rangle = 0$: First, differentiating $r^2 = \sum_{i=1}^n x_i^2$ yields

$$2r\, dr = \sum_{i=1}^{n} 2x_i\, dx_i, \tag{6.3}$$

which leads to

$$dr = \sum_{i=1}^{n} \frac{x_i}{r} dx_i = \sum_{i=1}^{n} \theta_i dx_i. \tag{6.4}$$

Since $\langle dx_i, dx_j\rangle = \delta_{ij}$, we get

$$\langle dx_k, dr\rangle = \left\langle dx_k, \sum_{i=1}^{n} \theta_i dx_i \right\rangle = \theta_k, \quad \text{and} \quad |dr|^2 = \langle dr, dr\rangle = \sum_{i=1}^{n} \theta_i^2 = 1. \tag{6.5}$$

Because $d\theta_k = d\left(\frac{x_k}{r}\right) = \frac{dx_k - \theta_k dr}{r}$, we get

$$\langle d\theta_k, dr\rangle = \frac{\langle dx_k, dr\rangle - \theta_k |dr|^2}{r} = \frac{\theta_k - \theta_k(1)}{r} = 0. \tag{6.6}$$

Further, we have

$$\sum_{k=1}^{n} \theta_k^2 = \sum_{k=1}^{n} \frac{x_k^2}{r^2} = \frac{r^2}{r^2} = 1. \tag{6.7}$$

The norm of the gradient satisfies $\displaystyle\sum_{k=1}^{n} |d\theta_k|^2 = \frac{n-1}{r^2}$. Using $|dx_k|^2 = 1$ and $\langle dx_k, dr\rangle = \theta_k$:

$$|d\theta_k|^2 = \frac{|dx_k|^2 - 2\theta_k\langle dx_k, dr\rangle + \theta_k^2|dr|^2}{r^2} = \frac{1 - 2\theta_k^2 + \theta_k^2}{r^2} = \frac{1 - \theta_k^2}{r^2}. \tag{6.8}$$

Summing over $k = 1, \dots, n$:

$$\sum_{k=1}^{n} |d\theta_k|^2 = \frac{\sum_{k=1}^{n} 1 - \sum_{k=1}^{n} \theta_k^2}{r^2} = \frac{n-1}{r^2}. \tag{6.9}$$

Let $a = a(t)$ be a smooth function, and let a' denote the differentiation with respect to t. Using the chain rule with $\partial_r t = -1/r$:

$$\partial_r a = a'(t) \cdot \partial_r t = -\frac{a'}{r}. \tag{6.10}$$

Differentiating again with respect to r leads to:

$$\partial_{rr} a = \partial_r \left(-\frac{a'}{r} \right) = \frac{a'}{r^2} - \frac{\partial_r(a')}{r} = \frac{a'}{r^2} - \frac{a'' \cdot (-1/r)}{r} = \frac{a'' + a'}{r^2}. \tag{6.11}$$

Consequently, the Laplacian in spherical coordinates is given by

$$\Delta a = \partial_{rr} a + \frac{n-1}{r} \partial_r a = \frac{a'' + a'}{r^2} + \frac{n-1}{r} \left(-\frac{a'}{r} \right) = \frac{1}{r^2} \left[a'' - (n-2)a' \right]. \tag{6.12}$$

Moreover, we have:

$$d(a\theta_k) = a d\theta_k + \theta_k da = a d\theta_k + \theta_k a' dt = a d\theta_k - a' \theta_k \frac{dr}{r}. \tag{6.13}$$

Using $\Delta_{S^{n-1}} \theta_k = -(n-1)\theta_k$, and the fact that ∇a is purely radial while $\nabla \theta_k$ is tangential (so $\langle \nabla a, \nabla \theta_k \rangle = 0$), we get

$$\Delta(a\theta_k) = (\Delta a)\theta_k + a(\Delta \theta_k) = \frac{1}{r^2} \left[a'' - (n-2)a' - (n-1)a \right] \theta_k. \tag{6.14}$$

Step 2: Definition of u.
Set

$$\lambda := n - 2 \tag{6.15}$$

and define, for $t \geq T$,

$$h(t) := \frac{e^{\lambda(t-T)}}{t}, \quad g(t) := \frac{e^{\lambda(t-T)}}{t^{3/2}}. \tag{6.16}$$

Because $e^{t-T} = e^{\log(1/r)} = r^{-1}$ (see (6.2)), these functions can be written equivalently as

$$h(t) = \frac{r^{2-n}}{t}, \quad g(t) = \frac{r^{2-n}}{t^{3/2}}. \tag{6.17}$$

Define for $x \neq 0$

$$u(x) := (h(t), g(t)\theta_1, \dots, g(t)\theta_n) \in \mathbb{R} \oplus \mathbb{R}^n = \mathbb{R}^{n+1} \tag{6.18}$$

and assign an arbitrary value to $u(0)$.

We explicitly compute the derivatives of $h(t)$ and $g(t)$:

$$h'(t) = \frac{\lambda e^{\lambda(t-T)} t - e^{\lambda(t-T)}}{t^2} = h(t)\left(\lambda - \frac{1}{t}\right), \tag{6.19}$$

$$h''(t) = h'(t)\left(\lambda - \frac{1}{t}\right) + h(t)\left(\frac{1}{t^2}\right) = h(t)\left(\lambda^2 - \frac{2\lambda}{t} + \frac{2}{t^2}\right), \tag{6.20}$$

$$g'(t) = \frac{\lambda e^{\lambda(t-T)} t^{3/2} - \frac{3}{2} t^{1/2} e^{\lambda(t-T)}}{t^3} = g(t)\left(\lambda - \frac{3}{2t}\right), \tag{6.21}$$

$$g''(t) = g'(t)\left(\lambda - \frac{3}{2t}\right) + g(t)\left(\frac{3}{2t^2}\right) = g(t)\left(\lambda^2 - \frac{3\lambda}{t} + \frac{15}{4t^2}\right). \tag{6.22}$$

Since $\lambda \geq 1$ (recall (6.15)) and $t \geq T \geq 2$, we have $\lambda - \frac{1}{t} \geq 1 - \frac{1}{2} = \frac{1}{2} > 0$, so

$$h'(t) = h(t)\left(\lambda - \frac{1}{t}\right) \geq \frac{1}{2} h(t) > 0. \tag{6.23}$$

Define now

$$F := \lambda h' - h'', \quad H := (n-1)g + \lambda g' - g''. \tag{6.24}$$

By (6.12) and (6.14) and recalling that $\lambda = n-2$, we have:

$$\begin{aligned} -\Delta u^0 &\overset{(6.18)}{=} -\Delta h \overset{(6.12)}{=} -\frac{h'' - \lambda h'}{r^2} \overset{(6.24)}{=} \frac{F}{r^2}, \\ -\Delta u^k &\overset{(6.18)}{=} -\Delta(g\theta_k) \overset{(6.14)}{=} -\frac{g'' - \lambda g' - (n-1)g}{r^2}\theta_k \overset{(6.24)}{=} \frac{H}{r^2}\theta_k, \quad k = 1, \ldots, n. \end{aligned} \tag{6.25}$$

Inserting (6.19)–(6.22) into the definition of F and H in (6.24) yields:

$$F = \lambda h\left(\lambda - \frac{1}{t}\right) - h\left(\lambda^2 - \frac{2\lambda}{t} + \frac{2}{t^2}\right) = h\left(\frac{\lambda}{t} - \frac{2}{t^2}\right), \tag{6.26}$$

$$H = (n-1)g + \lambda g\left(\lambda - \frac{3}{2t}\right) - g\left(\lambda^2 - \frac{3\lambda}{t} + \frac{15}{4t^2}\right) = g\left(n - 1 + \frac{3\lambda}{2t} - \frac{15}{4t^2}\right). \tag{6.27}$$

Step 3: Definition of the anti-symmetric potential Ω.
Set

$$D(t) := \frac{H(t)}{h'(t)}, \qquad C(t) := \frac{F(t) + g'(t)D(t)}{(n-1)g(t)}. \tag{6.28}$$

Since (recall equation (6.17))

$$g/h = t^{-1/2}, \tag{6.29}$$

the formulas (6.19)–(6.27) give explicitly:

$$D(t) \overset{(6.19)+(6.28)}{=} \frac{g(t)\left(n - 1 + \frac{3\lambda}{2t} - \frac{15}{4t^2}\right)}{h(t)\left(\lambda - \frac{1}{t}\right)} \overset{(6.17)}{=} t^{-1/2} \frac{n - 1 + \frac{3\lambda}{2t} - \frac{15}{4t^2}}{\lambda - \frac{1}{t}}, \tag{6.30}$$

and since $F(t) = g(t)\left(\lambda t^{-1/2} - 2t^{-3/2}\right)$ (see (6.26)) and using (6.21) we can rewrite C defined in (6.28) as:

$$C(t) = \frac{1}{n-1}\left[\lambda t^{-1/2} - 2t^{-3/2} + \left(\lambda - \frac{3}{2t}\right) D(t)\right]. \tag{6.31}$$

Since $t \geq T \geq 2$, the denominator $\lambda - 1/t \geq 1/2$ in equation (6.30) is bounded away from zero, and hence both functions D and C remain bounded by a constant depending only on n. Thus,

$$|C(t)| + |D(t)| \leq c_n t^{-1/2} \quad \text{for every } t \geq T \geq 2. \tag{6.32}$$

For $k = 1, \ldots, n$, introduce the scalar valued 1-forms

$$\omega_k := C(t) d\theta_k + D(t)\theta_k \frac{dr}{r}. \tag{6.33}$$

We define $\Omega = (\Omega_{ij})_{0 \leq i,j \leq n}$ by

$$\Omega_{0k} = \omega_k, \quad \Omega_{k0} = -\omega_k, \quad \Omega_{ij} = 0 \text{ otherwise}. \tag{6.34}$$

Thus, $\Omega_{ij} = -\Omega_{ji}$ by definition and hence $\Omega(x) \in \mathfrak{so}(n+1) \otimes \wedge^1$ for any $x \neq 0$.

Step 4: Equation (1.6) **holds pointwise a.e.**
Due to equation (6.13), we have

$$du^0 = -h' \frac{dr}{r}, \quad du^k = g d\theta_k - g' \theta_k \frac{dr}{r}. \tag{6.35}$$

We now use the following notation: $(\Omega \cdot \nabla u)^i = \sum_{k=0}^n \langle \Omega_{ik}, du^k \rangle$:
Case $i = 0$**:** Consider the inner product with du^k; using orthogonality $\langle d\theta_k, dr \rangle = 0$ gives:

$$\begin{aligned}(\Omega \cdot \nabla u)^0 &= \sum_{k=1}^n \left\langle C d\theta_k + D\theta_k \frac{dr}{r}, g d\theta_k - g' \theta_k \frac{dr}{r} \right\rangle \\ &= Cg \sum_{k=1}^n |d\theta_k|^2 - Dg' \sum_{k=1}^n \theta_k^2 \left|\frac{dr}{r}\right|^2 + 0 \\ &= Cg\left(\frac{n-1}{r^2}\right) - Dg'(1)\left(\frac{1}{r^2}\right) = \frac{(n-1)Cg - Dg'}{r^2}.\end{aligned} \tag{6.36}$$

By definition of $C(t)$ in (6.28),

$$(6.28) \quad \Leftrightarrow (n-1)Cg = F + g'D \quad \Leftrightarrow \quad (n-1)Cg - Dg' = F \tag{6.37}$$

holds equivalently.
Hence, we have

$$(\Omega \cdot \nabla u)^0 \overset{(6.36)}{=} \frac{F}{r^2} \overset{(6.25)}{=} -\Delta u^0. \tag{6.38}$$

Case $i \in \{1, \dots, n\}$: The only non-zero component is $\Omega_{i0} = -\omega_i$. Therefore:

$$
\begin{aligned}
(\Omega \cdot \nabla u)^i &= \langle -\omega_i, du^0 \rangle \overset{(6.33)}{=} \left\langle -Cd\theta_i - D\theta_i \frac{dr}{r}, -h' \frac{dr}{r} \right\rangle \\
&= Ch' \left\langle d\theta_i, \frac{dr}{r} \right\rangle + Dh'\theta_i \left| \frac{dr}{r} \right|^2 = 0 + \frac{Dh'}{r^2}\theta_i.
\end{aligned}
\tag{6.39}
$$

By definition $D = H/h'$, so $Dh' = H$. Thus:

$$
(\Omega \cdot \nabla u)^i = \frac{H}{r^2}\theta_i \overset{(6.25)}{=} -\Delta u^i. \tag{6.40}
$$

We have shown that

$$
-\Delta u = \Omega \cdot \nabla u \quad \text{on } B^n \setminus \{0\} \tag{6.41}
$$

holds pointwise.

Step 5: $\Omega \in L^n(\mathfrak{so}(n+1) \otimes \wedge^1)$.
By the computations of Step 1 together with the explicit definition in (6.33) we have:

$$
\sum_{k=1}^n |\omega_k|^2 = C^2 \sum_{k=1}^n |d\theta_k|^2 + D^2 \sum_{k=1}^n \theta_k^2 \left| \frac{dr}{r} \right|^2 \overset{(6.7)+(6.9)}{=} \frac{(n-1)C^2 + D^2}{r^2}. \tag{6.42}
$$

Since each ω_k appears twice in Ω (at the tuple $(0, k)$ and $(k, 0)$):

$$
|\Omega|^2 = \sum_{i,j=0}^n |\Omega_{ij}|^2 = 2 \sum_{k=1}^n |\omega_k|^2 \overset{(6.42)}{=} \frac{2((n-1)C^2 + D^2)}{r^2} \leq \frac{c_n}{r^2 t}, \tag{6.43}
$$

where we used estimate (6.32). Using spherical coordinates ($dx = r^{n-1} dr d\theta$), integrating $|\Omega|^n$ over B^n then yields

$$
\int_{B^n} |\Omega|^n \, dx \leq c_n |S^{n-1}| \int_0^1 \left(\frac{1}{r\sqrt{t}} \right)^n r^{n-1} dr = c_n |S^{n-1}| \int_0^1 \frac{dr}{r t^{n/2}}. \tag{6.44}
$$

Substituting $t = T - \log r$, $dt = -\frac{dr}{r}$ and $r = 1$ gives $t = T$, so $r \to 0$ gives $t \to \infty$ and hence

$$
\int_0^1 \frac{dr}{r t^{n/2}} = \int_T^\infty t^{-n/2} dt = \left[\frac{t^{1-n/2}}{1 - n/2} \right]_T^\infty = \frac{T^{1-n/2}}{\frac{n}{2} - 1} < \infty \quad \text{for } n \geq 3. \tag{6.45}
$$

Thus, $\|\Omega\|_{L^n(B^n)} \leq c_n T^{\frac{1}{n} - \frac{1}{2}} \to 0$ as $T \to \infty$. So for T large enough, we obtain $\|\Omega\|_{L^n(B^n)} < \varepsilon_0$.

Step 6. $u \in W^{1\frac{n}{n-1}}(B^n, \mathbb{R}^{n+1})$**.**
We have

$$
|\nabla u|^2 = |\nabla u^0|^2 + \sum_{k=1}^n |\nabla u^k|^2 \overset{(6.35)}{=} \frac{(h')^2 + (g')^2 + (n-1)g^2}{r^2}. \tag{6.46}
$$

Since $g = ht^{-1/2}$ (recall (6.17)), due to equations (6.19) and (6.21) we have

$$
(h')^2 + (g')^2 + (n-1)g^2 \approx \lambda^2 h^2, \tag{6.47}
$$

and hence

$$|\nabla u(x)| \sim \frac{h(t)}{r} \overset{(6.16)}{=} \frac{r^{2-n}}{rt} = \frac{1}{r^{n-1}t}. \tag{6.48}$$

Using $(n-1)\frac{n}{n-1} = n$, we get

$$\begin{aligned} \int_{B^n} |\nabla u|^{\frac{n}{n-1}} dx &\leq c_n |S^{n-1}| \int_0^1 r^{n-1} \left(\frac{1}{r^{n-1}t}\right)^{\frac{n}{n-1}} dr \\ &= c_n |S^{n-1}| \int_0^1 \frac{dr}{rt^{\frac{n}{n-1}}} = c_n |S^{n-1}| \int_T^\infty t^{-\frac{n}{n-1}} dt < \infty, \end{aligned} \tag{6.49}$$

since $\frac{n}{n-1} > 1$ for all $n \geq 3$.
The computation remains for the $L^{\frac{n}{n-1}}$–norm of u. Due to the definition of u in (6.18), we have

$$|u|^2 = h^2 + g^2 = h^2(1 + t^{-1}) \leq 2h^2 \text{ for } t \geq 2. \tag{6.50}$$

Thus:

$$\int_{B^n} |u|^{\frac{n}{n-1}} dx \overset{(6.17)}{\leq} c_n \int_0^1 r^{n-1} (r^{2-n} t^{-1})^{\frac{n}{n-1}} dr = c_n \int_0^1 r^{n-1-(n-2)p} t^{-\frac{n}{n-1}}\, dr. \tag{6.51}$$

For the radial exponent we have $n-1-(n-2)\frac{n}{n-1} = \frac{(n-1)^2 - n(n-2)}{n-1} = \frac{1}{n-1} > -1$, so the integrand is integrable near $r = 0$. Hence, $u \in W^{1\frac{n}{n-1}}(B^n, \mathbb{R}^{n+1})$.

Step 7: Removability of the singularity.
To verify that no Dirac mass is created at the origin, let $\varphi \in C_c^\infty(B^n, \mathbb{R}^{n+1})$. Integrating by parts on $B^n \setminus \overline{B_\rho}$ yields

$$\int_{B^n \setminus \overline{B_\rho}} \langle \nabla u, \nabla \varphi \rangle dx = \int_{B^n \setminus \overline{B_\rho}} (\Omega \cdot \nabla u) \cdot \varphi\, dx + \int_{\partial B_\rho} \partial_r u \cdot \varphi\, d\mathcal{H}^{n-1}. \tag{6.52}$$

By (6.35), we have $|\partial_r u| \leq \frac{|h'|+|g'|}{r}$. The integral over the boundary leads to

$$\int_{\partial B_\rho} |\partial_r u| d\mathcal{H}^{n-1} \leq c_n \rho^{n-1} \frac{|h'(t_\rho)| + |g'(t_\rho)|}{\rho} = c_n \rho^{n-2} \left(|h'(t_\rho)| + |g'(t_\rho)|\right), \tag{6.53}$$

where $t_\rho = T + \log(1/\rho)$.
Recalling that $e^{\lambda(t_\rho - T)} = e^{\lambda \log(1/\rho)} = \rho^{-\lambda} = \rho^{-(n-2)}$, we substitute $h'(t_\rho)$ and $g'(t_\rho)$ from (6.19) and (6.21) in order to get:

$$h'(t_\rho) = \rho^{-(n-2)} t_\rho^{-1} \left(\lambda - \frac{1}{t_\rho}\right), \quad g'(t_\rho) = \rho^{-(n-2)} t_\rho^{-3/2} \left(\lambda - \frac{3}{2t_\rho}\right). \tag{6.54}$$

Multiplying by ρ^{n-2} then gives

$$\rho^{n-2} h'(t_\rho) = t_\rho^{-1} \left(\lambda - \frac{1}{t_\rho}\right), \quad \rho^{n-2} g'(t_\rho) = t_\rho^{-3/2} \left(\lambda - \frac{3}{2t_\rho}\right). \tag{6.55}$$

As $\rho \to 0^+$, $t_\rho = T + \log(1/\rho) \to +\infty$ and thus:

$$\lim_{\rho\to 0}\int_{\partial B_\rho} |\partial_r u| d\mathcal{H}^{n-1} \le c_n \lim_{t_\rho\to\infty}\left(\frac{\lambda - \frac{1}{t_\rho}}{t_\rho} + \frac{\lambda - \frac{3}{2t_\rho}}{t_\rho^{3/2}}\right) = 0. \tag{6.56}$$

This shows that $\rho \to 0$ yields

$$\int_{B^n} \langle \nabla u, \nabla \varphi\rangle dx = \int_{B^n} (\Omega \cdot \nabla u)\cdot \varphi\, dx, \tag{6.57}$$

and we conclude that $-\Delta u = \Omega \cdot \nabla u$ holds in $\mathcal{D}'(B^n)$.

Step 8: $u \notin L^\infty$ and is therefore not continuous.
Recall the definition of the first component of u (see (6.18)):

$$u^0(x) = h(t) = \frac{|x|^{2-n}}{T + \log(1/|x|)}. \tag{6.58}$$

Since $n \ge 3$, as $x \to 0$, $|x|^{2-n}$ grows polynomially, $T + \log(1/|x|)$ grows logarithmically. Hence,

$$u^0(x) \to +\infty \quad \text{as } x \to 0. \tag{6.59}$$

Thus, $u \notin L^\infty$ and cannot have a continuous representative. □

Remark 6.2. *The construction above has the exact asymptotic behaviour:*

$$|\Omega(x)| \sim \frac{1}{|x|(T+\log(1/|x|))^{1/2}}, \quad |\nabla u(x)| \sim \frac{1}{|x|^{n-1}(T+\log(1/|x|))}. \tag{6.60}$$

By standard rearrangement characterisation of Lorentz spaces, we have

$$\Omega \in \left(\bigcap_{q>2} L^{(n,q)}(B^n)\right) \setminus L^{(n,2)}(B^n), \tag{6.61}$$

whereas

$$\nabla u \in \left(\bigcap_{s>1} L^{(\frac{n}{n-1},s)}(B^n)\right) \setminus L^{(\frac{n}{n-1},1)}(B^n). \tag{6.62}$$

Thus, the result is optimal.

Appendix A.

Let $1 \le q \le 2$ and denote by $2 \le q'$ its corresponding conjugate exponent. In the following, let $\frac{n}{2} < p < n$. Then we have $W^{1,(p,1)} \hookrightarrow L^{(s,1)} \subset L^{(\frac{np}{n-p},q')}$ for any $s \le \frac{np}{n-p}$ and hence $n < \frac{np}{n-p}$ is satisfied by the restriction on the parameter p. Further, since $p < n$ we have

$$\frac{1}{p} = \frac{1}{n} + \frac{1}{\frac{np}{n-p}}, \quad \text{and} \quad 1 = \frac{1}{q} + \frac{1}{q'}, \tag{A.1}$$

which for any $f \in L^{(n,q)}$ and $g \in W^{1,(p,1)}$ leads to

$$\|f\cdot g\|_{L^{(p,1)}} \lesssim \|f\|_{L^{(n,q)}}\|g\|_{L^{(\frac{np}{n-p},q')}} \lesssim \|f\|_{L^{(n,q)}}\|g\|_{W^{1,(p,1)}}. \tag{A.2}$$

Lemma A.1. *Assume there exists* $\varepsilon > 0$ *and a constant* $C > 0$ *such that for any* $\Omega \in W^{1,(p,1)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ *satisfying*

$$\|\Omega\|_{L^{(n,q)}} < \varepsilon(n,m), \tag{A.3}$$

there exists $P \in W^{2,(p,1)}(B^n, \mathrm{SO}(m))$ *and* $\xi \in W^{2,(p,1)}(B^n, \mathfrak{so}(m) \otimes \bigwedge^{n-2})$ *such that*

$$\begin{cases} *d\xi = P^{-1}dP + P^{-1}\Omega P & \text{in } B^n, \\ d^*\xi = 0 & \text{in } B^n, \\ \xi = 0 & \text{on } \partial B^n, \end{cases} \tag{A.4}$$

and satisfy

$$\|dP\|_{W^{1,(p,1)}} + \|d\xi\|_{W^{1,(p,1)}} \leq C\|\Omega\|_{W^{1,(p,1)}} \tag{A.5}$$

and

$$\|dP\|_{L^{(n,q)}} + \|d\xi\|_{L^{(n,q)}} \leq C\|\Omega\|_{L^{(n,q)}} \leq C\varepsilon(n,m). \tag{A.6}$$

Proof of Lemma A.1. We follow the continuity method by Uhlenbeck [24] with the techniques established by Rivière [18].
Introduce for any constants $\varepsilon > 0$ and $C \geq 0$ the set

$$\mathcal{U}_{\varepsilon,C} := \left\{ \begin{array}{l} \Omega \in W^{1,(p,1)}(B^n, \mathfrak{so}(m) \otimes \wedge^1(\mathbb{R}^n)) : \ \|\Omega\|_{L^{(n,q)}} \leq \varepsilon, \text{ and} \\ \text{there exist } P \text{ and } \xi \text{ satisfying (A.4), (A.5), (A.6).} \end{array} \right\} \tag{A.7}$$

Further, define the set

$$\mathcal{V}_\varepsilon := \left\{ \Omega \in W^{1,(p,1)}(B^n, \mathfrak{so}(m) \otimes \wedge^1) : \ \|\Omega\|_{L^{(n,q)}} \leq \varepsilon \right\}. \tag{A.8}$$

We show that for $\varepsilon > 0$ small enough and $C \geq 0$ large enough, $\mathcal{V}_\varepsilon = \mathcal{U}_{\varepsilon,C}$ by the following steps:

1. Step: $\mathcal{U}_{\varepsilon,C}$ is non-empty ($0 \in \mathcal{U}_{\varepsilon,C}$).
2. Step: $\mathcal{V}_\varepsilon$ is path-connected.
3. Step: $\mathcal{U}_{\varepsilon,C}$ is closed in $\mathcal{V}_\varepsilon$ with respect to the $W^{1,(p,1)}$-topology.
4. Step: $\mathcal{U}_{\varepsilon,C}$ is open in $\mathcal{V}_\varepsilon$ with respect to the $W^{1,(p,1)}$-topology.

1. Step: $\mathcal{U}_{\varepsilon,C}$ is non-empty.
Trivially $0 \in \mathcal{U}_{\varepsilon,C}$ with the corresponding solution

$$P = \mathrm{Id}_m \in W^{2,(p,1)}, \qquad \xi = 0 \in W^{2,(p,1)}.$$

2. Step: $\mathcal{V}_\varepsilon$ is path-connected.
We show that $\mathcal{V}_\varepsilon$ is star-shaped and hence path-connected. Fix $\Omega_0 \in \mathcal{V}_\varepsilon$ and let $\Omega \in \mathcal{V}_\varepsilon$ be arbitrary. Define the path by

$$\Omega_t := t\Omega + (1-t)\Omega_0, \qquad t \in [0,1].$$

Then, for any $t \in [0,1]$, triangle inequality gives

$$\|\Omega_t\|_{L^{(n,q)}} \le t\|\Omega\|_{L^{(n,q)}} + (1-t)\|\Omega_0\|_{L^{(n,q)}} \le t\varepsilon + (1-t)\varepsilon = \varepsilon,$$

and hence $\Omega_t \in \mathcal{V}_\varepsilon$.

3. Step: $\mathcal{U}_{\varepsilon,C}$ is closed with respect to the $W^{1,(p,1)}$-topology.
Let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{U}_{\varepsilon,C}$ with

$$\Omega_k \to \Omega_\infty \quad \text{strongly in } W^{1,(p,1)},$$

where the limit satisfies $\Omega_\infty \in \mathcal{V}_\varepsilon$. We claim that $\Omega_\infty \in \mathcal{U}_{\varepsilon,C}$.
Since $\Omega_k \in \mathcal{U}_{\varepsilon,C}$, there exists a sequence of solutions to (A.4), denoted by $P_k \in W^{2,(p,1)}(B^n, \mathrm{SO}(m))$ and $\xi_k \in W^{2,(p,1)}(B^n, \mathfrak{so}(m) \otimes \bigwedge^{n-2})$ for each $k \in \mathbb{N}$. Due to the uniform bounds $\|\Omega_k\|_{L^{(n,q)}} \le \varepsilon$ and $\|\Omega_k\|_{W^{1,(p,1)}} \lesssim \|\Omega_\infty\|_{W^{1,(p,1)}} < \infty$, together with estimates (A.5) and (A.6), yields a weakly convergent subsequence (P_{k_j}, ξ_{k_j}) such that

$$P_{k_j} \rightharpoonup P_\infty \quad \text{and} \quad \xi_{k_j} \rightharpoonup \xi_\infty \quad \text{weakly in } W^{2,(p,1)}. \tag{A.9}$$

By lower semicontinuity of the Sobolev-Lorentz norms, the weak limits satisfy

$$\|dP_\infty\|_{W^{1,(p,1)}} + \|d\xi_\infty\|_{W^{1,(p,1)}} \le \liminf_{j\to\infty} C\|\Omega_{k_j}\|_{W^{1,(p,1)}} = C\|\Omega_\infty\|_{W^{1,(p,1)}}, \tag{A.10}$$

and

$$\|dP_\infty\|_{L^{(n,q)}} + \|d\xi_\infty\|_{L^{(n,q)}} \le \liminf_{j\to\infty} C\|\Omega_{k_j}\|_{L^{(n,q)}} = C\|\Omega_\infty\|_{L^{(n,q)}}. \tag{A.11}$$

It remains to show that (P_∞, ξ_∞) solves (A.4) with respect to Ω_∞ and that $P_\infty \in \mathrm{SO}(m)$ almost everywhere.
First, by Rellich-Kondrachov compact embedding, $W^{2,(p,1)}(B^n) \hookrightarrow W^{1,(s,1)}(B^n)$ for any $1 \le s < \frac{np}{n-p}$, there exists a strongly convergent subsequence in $L^{(s,1)}$, which implies pointwise convergence almost everywhere of $P_{k_j} \to P_\infty$ and $d^*\xi_{k_j} \to d^*\xi_\infty$. Passing to the limit in $P_{k_j}P_{k_j}^\top = \mathrm{Id}_m$ and $d^*\xi_{k_j} = 0$ pointwise a.e. confirms that $P_\infty \in W^{2,(p,1)}(B^n, \mathrm{SO}(m))$ and $d^*\xi_\infty = 0$ on B^n almost everywhere.

Second, we show that the limit is a solution in the sense of distributions, i.e.,

$$*d\xi_\infty = P_\infty^{-1}dP_\infty + P_\infty^{-1}\Omega_\infty P_\infty \quad \text{in } \mathcal{D}'(B^n). \tag{A.12}$$

The compact embedding $W^{2,(p,1)}(B^n) \hookrightarrow W^{1,s}(B^n)$ ensures strong convergence:

$$P_{k_j} \to P_\infty \quad \text{in } L^{(s,1)}(B^n) \quad \text{and} \quad dP_{k_j} \to dP_\infty \quad \text{strongly in } L^{(s,1)}(B^n) \quad \forall 1 \le s < \frac{np}{n-p}.$$

Left-Hand Side (LHS) of (A.12)*:* For any test function $\Phi \in C_c^\infty(B^n, \mathfrak{so}(m) \otimes \wedge^1(\mathbb{R}^n))$, weak convergence $\xi_{k_j} \rightharpoonup \xi_\infty$ in $W^{2,(p,1)}$ implies $d\xi_{k_j} \rightharpoonup d\xi_\infty$ weakly in $W^{1,(p,1)}$ and thus in $L^{(n,q)}$:

$$\left|\int_{B^n} (*d\xi_{k_j} - *d\xi_\infty) \cdot \Phi\right| \longrightarrow 0. \tag{A.13}$$

Right-Hand Side (RHS) of (A.12)*:* Testing against $\Phi \in C_c^\infty(B^n, \mathfrak{so}(m) \otimes \wedge^1(\mathbb{R}^n))$, we bound each term:

$$\begin{aligned}
&\left|\int_{B^n} (P_\infty^{-1} dP_\infty - P_{k_j}^{-1} dP_{k_j})\Phi\right| \\
&\leq \left|\int_{B^n} P_\infty^{-1}(dP_\infty - dP_{k_j})\Phi\right| + \left|\int_{B^n} (P_{k_j}^{-1} - P_\infty^{-1}) dP_{k_j}\Phi\right| \\
&\lesssim \left|\int_{B^n} (dP_\infty - dP_{k_j})(P_\infty^{-\top}\Phi)\right| + \|dP_{k_j}\|_{L^{(n,q)}} \|P_{k_j}^{-1} - P_\infty^{-1}\|_{L^{(\frac{n}{n-1},q')}} \|\Phi\|_{L^\infty} \\
&\longrightarrow 0,
\end{aligned} \tag{A.14}$$

and similarly

$$\begin{aligned}
&\left|\int_{B^n} (P_\infty^{-1}\Omega_\infty P_\infty - P_{k_j}^{-1}\Omega_{k_j} P_{k_j})\Phi\right| \\
&\leq \left|\int_{B^n} P_\infty^{-1}\Omega_\infty(P_\infty - P_{k_j})\Phi\right| + \left|\int_{B^n} (P_{k_j}^{-1}\Omega_{k_j} - P_\infty^{-1}\Omega_\infty) P_{k_j}\Phi\right| \\
&\lesssim \|P_\infty^{-1}\|_{L^\infty} \|\Omega_\infty\|_{L^{(n,q)}} \|P_\infty - P_{k_j}\|_{L^{(\frac{n}{n-1},q')}} \|\Phi\|_{L^\infty} \\
&\quad + \|P_{k_j}\|_{L^\infty} \Big(\|P_{k_j}^{-1}\|_{L^\infty} \|(\Omega_{k_j} - \Omega_\infty)\Phi\|_{L^1} + \|P_{k_j}^{-1} - P_\infty^{-1}\|_{L^{(\frac{n}{n-1},q')}} \|\Omega_\infty\|_{L^{(n,q)}} \|\Phi\|_{L^\infty} \Big) \\
&\longrightarrow 0.
\end{aligned} \tag{A.15}$$

Using the trace theorem for Sobolev–Lorentz spaces, the continuous embedding

$$W^{2,(p,1)}(B^n) \hookrightarrow W^{2-\frac{1}{p},p}(\partial B^n) \hookrightarrow L^1(\partial B^n) \tag{A.16}$$

ensures that $\xi_k\big|_{\partial B^n} = 0$ passes to the weak limit, so $\xi_\infty\big|_{\partial B^n} = 0$.
Hence, Ω_∞ fulfils each condition for belonging to $\mathcal{U}_{\varepsilon,C}$.

4. Step: $\mathcal{U}_{\varepsilon,C}$ is open in $\mathcal{V}_\varepsilon$ with respect to the $W^{1,(p,1)}$-topology.
In this step, we need to restrict our choice of $\varepsilon > 0$, choosing it sufficiently small, and set $C \geq 0$ to be sufficiently large.
Let $\Omega_0 \in \mathcal{U}_{\varepsilon,C}$. We claim that there exists $\delta > 0$ such that for any $\omega \in W^{1,(p,1)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$

$$\text{with } \|\omega\|_{W^{1,(p,1)}} < \delta, \qquad \text{we have} \qquad \Omega_0 + \omega \in \mathcal{U}_{\varepsilon,C}. \tag{A.17}$$

Since $\Omega_0 \in \mathcal{U}_{\varepsilon,C}$, there exists a corresponding pair of solutions (P_0, ξ_0) satisfying (A.4). Define the non-linear operator

$$\begin{aligned}
\mathcal{N}\colon W^{1,(p,1)}(B^n, \mathfrak{so}(m) \otimes \wedge^1) \times X &\to L^{(p,1)}(B^n, \mathfrak{so}(m)) \times W^{1-\frac{1}{p},(p,1)}(\partial B^n, \mathfrak{so}(m)), \\
(\lambda, U) &\mapsto \left(d^*\Big(e^{-U} de^U + e^{-U}(*d\xi_0 + \lambda)e^U\Big),\ \partial_r U\Big|_{\partial B^n} \right),
\end{aligned} \tag{A.18}$$

where

$$\begin{aligned}
X &:= \left\{ U \in W^{2,(p,1)}(B^n, \mathfrak{so}(m)) \;\middle|\; \int_{B^n} U\, dx = 0 \right\} \\
Y &:= L^{(p,1)}(B^n, \mathfrak{so}(m)) \times W^{1-\frac{1}{p},(p,1)}(\partial B^n, \mathfrak{so}(m)).
\end{aligned} \tag{A.19}$$

Since $P \in W^{2,(p,1)}(B^n, \mathrm{SO}(m))$, the map $\lambda \mapsto P^{-1}\lambda P$ and its inverse $\zeta \mapsto P\zeta P^{-1}$ are continuous from $W^{1,(p,1)}$ to $W^{1,(p,1)}$. Indeed, because $P(x) \in \mathrm{SO}(m)$ pointwise, we have

$$\|P\lambda P^{-1}\|_{L^{(p,1)}} \lesssim \|\lambda\|_{L^p}, \tag{A.20}$$

and applying the product rule yields:

$$\begin{aligned} \|\nabla(P\lambda P^{-1})\|_{L^{(p,1)}} &\lesssim \|\nabla P\lambda P^{-1}\|_{L^{(p,1)}} + \|P\nabla\lambda P^{-1}\|_{L^{(p,1)}} + \|P\lambda\nabla P^{-1}\|_{L^{(p,1)}} \\ &\lesssim \|\nabla P\|_{L^{(n,q)}}\|\lambda\|_{L^{(\frac{np}{n-p},q')}} + \|\nabla\lambda\|_{L^{(p,1)}} + \|\nabla P\|_{L^{(n,q)}}\|\lambda\|_{L^{(\frac{np}{n-p},q')}} \\ &\lesssim \|\nabla P\|_{L^{(n,q)}}\|\lambda\|_{W^{1,(p,1)}} + \|\nabla\lambda\|_{L^{(p,1)}}, \end{aligned} \tag{A.21}$$

where we used $W^{1,(p,1)}(B^n) \hookrightarrow L^{(\frac{np}{n-p},q')}(B^n)$.
Thus, the non-linear map $\mathcal{N}$ is C^1-smooth. We follow the strategy of Lemma 2.7 and Lemma 2.8 in [24].
To derive the linearized operator $L_1 := \partial_U\mathcal{N}(0,0)\colon X \to Y$, we compute the directional derivative of $\mathcal{N}(0,tV)$ at $t = 0$ for $V \in X$. Expanding the matrix exponentials yields

$$\begin{aligned} e^{tV} &= \mathrm{Id}_m + tV + \mathcal{O}(t^2), \\ e^{-tV} &= \mathrm{Id}_m - tV + \mathcal{O}(t^2), \\ d(e^{tV}) &= t\,dV + \mathcal{O}(t^2). \end{aligned} \tag{A.22}$$

Substituting these expansions into the first component of $\mathcal{N}$ gives

$$\begin{aligned} e^{-tV}d(e^{tV}) + e^{-tV} * d\xi_0 e^{tV} &= (\mathrm{Id}_m - tV)(t\,dV) + (\mathrm{Id}_m - tV) * d\xi_0(\mathrm{Id}_m + tV) + \mathcal{O}(t^2) \\ &= *d\xi_0 + t\Big(dV + *d\xi_0 V - V * d\xi_0\Big) + \mathcal{O}(t^2). \end{aligned} \tag{A.23}$$

Differentiating with respect to t at $t = 0$ yields

$$\frac{d}{dt}\bigg|_{t=0}\Big(e^{-tV}d(e^{tV}) + e^{-tV} * d\xi_0 e^{tV}\Big) = dV + [*d\xi_0, V]. \tag{A.24}$$

Applying d^* (noting that $d^*dV = \Delta V$), the linearized operator $L_1 := \partial_U\mathcal{N}(0,0)\colon X \to Y$ is explicitly given by

$$L_1 V = \Big(\Delta V + d^*(*d\xi_0 V - V * d\xi_0),\ \partial_r V\Big|_{\partial B^n}\Big). \tag{A.25}$$

Standard elliptic theory for the Laplacian $L_0 V := (\Delta V, \partial_r V|_{\partial B^n})$ in Sobolev–Lorentz spaces yields

$$\|V\|_{W^{2,(p,1)}(B^n)} \le c_0\,\|L_0 V\|_Y = c_0\Big(\|\Delta V\|_{L^{(p,1)}(B^n)} + \|\partial_r V\|_{W^{1-\frac{1}{p},(p,1)}(\partial B^n)}\Big). \tag{A.26}$$

Combining L_0 and L_1 via the triangle inequality gives

$$\|V\|_{W^{2,(p,1)}(B^n)} \le c_0\|L_1 V\|_Y + c_0\|d\xi_0 \wedge dV - dV \wedge d\xi_0\|_{L^{(p,1)}(B^n)}. \tag{A.27}$$

It remains to estimate the perturbation term. Using $W^{1,(p,1)}(B^n) \hookrightarrow L^{(\frac{np}{n-p},q')}(B^n)$ (see (A.2)), yields

$$\|d\xi_0 \wedge dV\|_{L^{(p,1)}} \leq \|d\xi_0\|_{L^{(n,q)}} \|\nabla V\|_{L^{(\frac{np}{n-p},q')}} \leq C_2 \|d\xi_0\|_{L^{(n,q)}} \|V\|_{W^{2,(p,1)}}. \tag{A.28}$$

Inserting (A.28) in (A.26) and using the anti-symmetry of the wedge-product, we obtain:

$$\|V\|_{W^{2,(p,1)}(B^n)} \leq c_0 \|L_1 V\|_Y + c_0 C_3 \|d\xi_0\|_{L^{(n,q)}(B^n)} \|V\|_{W^{2,(p,1)}(B^n)}. \tag{A.29}$$

Since ξ_0 is a solution of $\Omega_0 \in \mathcal{U}_{\varepsilon,C}$, we have $\|d\xi_0\|_{L^{(n,q)}(B^n)} \leq \varepsilon$ and choosing $\varepsilon < \frac{1}{2c_0 C_3}$ small enough, we absorb the perturbation term into the left-hand side and get:

$$\|V\|_{W^{2,(p,1)}(B^n)} \leq 2c_0 \|L_1 V\|_Y. \tag{A.30}$$

If $L_1 V = 0$, estimate (A.30) immediately implies $\|V\|_{W^{2,(p,1)}} = 0$, so $V = 0$. Thus, $\ker(L_1) = \{0\}$.
It remains to show that L_1 is surjective and hence an isomorphism. Recall that $L_0 \colon X \to Y$ is a classical elliptic operator defining an isomorphism on X due to the mean-zero assumption. The family $(L_t)_{t\in[0,1]}$, where

$$t \mapsto L_t := L_0 + t\,K, \qquad \text{and} \;\; K(V) = \big(*\,(d\xi_0 \wedge dV) - *(dV \wedge d\xi_0),\ 0\big) \tag{A.31}$$

defines a continuous path of bounded operators. For ε small enough, L_t remains Fredholm for all $t \in [0,1]$. By the invariance of the Fredholm index under continuous perturbations, we obtain

$$\operatorname{ind}(L_1) = \operatorname{ind}(L_0) = 0. \tag{A.32}$$

By the definition of the index and since $\ker(L_1) = \{0\}$, it follows that L_1 is surjective.

Applying the Implicit Function Theorem gives a $\delta > 0$ and an open neighborhood $\mathcal{O}$ of 0 in X such that for every $\lambda \in W^{1,(p,1)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ satisfying $\|\lambda\|_{W^{1,(p,1)}(B^n)} < \delta$, there exists a unique $U_\lambda \in \mathcal{O}$ with $\mathcal{N}(\lambda, U_\lambda) = 0$ and $\int_{B^n} U_\lambda \, dx = 0$.

Since the map $\lambda \mapsto U_\lambda$ is C^1-continuous with $U_0 = 0$, we have $\|U_\lambda\|_{W^{2,(p,1)}} \to 0$ as $\|\lambda\|_{W^{1,(p,1)}} \to 0$. Because $W^{2,(p,1)}(B^n)$ is a Banach algebra under matrix multiplication, setting $Q_\lambda := e^{U_\lambda}$ yields

$$\|d(Q_\lambda - \mathrm{Id}_m)\|_{W^{1,(p,1)}} \leq C \|dU_\lambda\|_{W^{1,(p,1)}} < \alpha, \tag{A.33}$$

for any given $\alpha > 0$, assuming that δ is chosen sufficiently small.

Since (P_0, ξ_0) solves (A.4) for Ω_0, we define $P_\lambda := P_0 Q_\lambda$ and obtain:

$$\begin{aligned} *d\xi_\lambda &= Q_\lambda^{-1} dQ_\lambda + Q_\lambda^{-1}(P_0^{-1} dP_0 + P_0^{-1}\Omega_0 P_0 + \lambda) Q_\lambda \\ &= (P_0 Q_\lambda)^{-1} d(P_0 Q_\lambda) + (P_0 Q_\lambda)^{-1}(\Omega_0 + P_0 \lambda P_0^{-1})(P_0 Q_\lambda) \\ &= P_\lambda^{-1} dP_\lambda + P_\lambda^{-1}(\Omega_0 + \omega) P_\lambda, \end{aligned} \tag{A.34}$$

where $\omega \coloneqq P_0 \lambda P_0^{-1}$. Thus, (P_λ, ξ_λ) solves (A.4) for $\Omega := \Omega_0 + \omega$, with

$$\|\omega\|_{W^{1,(p,1)}} \leq C\|\lambda\|_{W^{1,(p,1)}} < \delta. \tag{A.35}$$

Moreover, standard elliptic estimates applied to

$$*d(\xi_\lambda - \xi_0) = Q_\lambda^{-1} dQ_\lambda + Q_\lambda^{-1}\big[* d\xi_0, Q_\lambda - \mathrm{Id}_m\big] + Q_\lambda^{-1}\lambda Q_\lambda$$

along with $d^*(\xi_\lambda - \xi_0) = 0$ and $\xi_\lambda - \xi_0 = 0$ on ∂B^n using (A.33) and (A.35) yields that:

$$\|d(\xi_\lambda - \xi_0)\|_{W^{1,(p,1)}} \lesssim \|d(\xi_\lambda - \xi_0)\|_{W^{1,(p,1)}} \lesssim \alpha + \delta =: \beta. \tag{A.36}$$

Note, that (A.33) immediately implies

$$\|d(P_\lambda - P_0)\|_{W^{1,(p,1)}} \leq C\|d(Q_\lambda - \mathrm{Id}_m)\|_{W^{1,(p,1)}} < \alpha. \tag{A.37}$$

Choosing $\varepsilon, \alpha, \beta > 0$ sufficiently small such that $C\varepsilon + \alpha + \beta \leq \delta$, the triangle inequality yields

$$\begin{aligned}\|dP_\lambda\|_{L^{(n,q)}} + \|d\xi_\lambda\|_{L^{(n,q)}} &\leq \|dP_0\|_{L^{(n,q)}} + \|d\xi_0\|_{L^{(n,q)}} \\ &\qquad + \|d(P_\lambda - P_0)\|_{L^{(n,q)}} + \|d(\xi_\lambda - \xi_0)\|_{L^{(n,q)}} \\ &\leq C\varepsilon + \alpha + \beta \leq \delta.\end{aligned} \tag{A.38}$$

Therefore, (P_λ, ξ_λ) satisfies the assumption of Lemma A.2, which implies the a-priori bounds (A.5) and (A.6) for (P_λ, ξ_λ).
This proves $\Omega_0 + \omega \in \mathcal{U}_{\varepsilon,C}$, and hence that $\mathcal{U}_{\varepsilon,C}$ is open in $\mathcal{V}_\varepsilon$.
Conclusion. Step 1 – Step 4 implies $\mathcal{U}_{\varepsilon,C} = \mathcal{V}_\varepsilon$ and this finishes the proof of Lemma A.1. □

It remains to prove the following auxiliary Lemma.

Lemma A.2. *There exist $C(n) > 0$, $\varepsilon > 0$ and $\delta > 0$ such that for every $P \in W^{2,(p,1)}(B^n, \mathrm{SO}(m))$ and $\xi \in W^{2,(p,1)}(B^n, \mathfrak{so}(m) \otimes \wedge^{n-2})$ satisfying* (A.4) *for some $\Omega \in W^{1,(p,1)}(B^n, \mathfrak{so}(m) \otimes \wedge^1)$ with $\|\Omega\|_{L^{(n,q)}} < \varepsilon$, if*

$$\|dP\|_{L^{(n,q)}} + \|d\xi\|_{L^{(n,q)}} \leq \delta,$$

then (A.5) *and* (A.6) *are satisfied.*

Proof. We decompose $\xi = v + w$, where $v, w \in W^{2,(p,1)}(B^n, \wedge^{n-2})$ satisfy:

$$\begin{cases} \Delta v = *(dP^{-1} \wedge dP) & \text{in } B^n, \\ v = 0 & \text{on } \partial B^n, \end{cases} \qquad \begin{cases} \Delta w = *d(P^{-1}\Omega P) & \text{in } B^n, \\ w = 0 & \text{on } \partial B^n. \end{cases} \tag{A.39}$$

Estimates containing v:

$L^{(n,q)}$ *Bound for dv:* In order to exploit the div-curl structure, we use that $dv \in W^{1,(p,1)} \hookrightarrow L^{(n,1)} \hookrightarrow L^{(n,q)}$ (for $1 \leq q \leq 2$) and hence

$$\begin{aligned} \|dv\|_{L^{(n,q)}(B^n)} &\lesssim \|dv\|_{L^{(n,1)}(B^n)} \lesssim \|\Delta v\|_{L^{(\frac{n}{2},1)}(B^n)} \lesssim \|dP^{-1}\|_{L^{(n,q)}} \|dP\|_{L^{(n,q')}} \\ &\overset{L^{(n,q)} \subset L^{(n,q')}}{\leq} C\delta \|dP\|_{L^{(n,q)}(B^n)}. \end{aligned} \tag{A.40}$$

$W^{1,(p,1)}$ *Bound for dv:* Using Calderón-Zygmund estimates together with the Lorentz-Hölder inequality and the Sobolev embedding $W^{1,(p,1)}(B^n) \hookrightarrow L^{(\frac{np}{n-p},1)}(B^n) \hookrightarrow L^{(\frac{np}{n-p},q')}(B^n)$, we get

$$\|dv\|_{W^{1,(p,1)}} \lesssim \|\Delta v\|_{L^{(p,1)}} \lesssim \|dP\|_{L^{(n,q)}} \|dP\|_{L^{(\frac{np}{n-p},q')}} \lesssim \delta \|dP\|_{W^{1,(p,1)}}. \tag{A.41}$$

Estimates containing w:

$L^{(n,q)}$ *Bound for dw:* Again by standard elliptic estimates and $|P| \leq 1$ we have

$$\|dw\|_{L^{(n,q)}} \leq C\|P^{-1}\Omega P\|_{L^{(n,q)}} \leq C\|\Omega\|_{L^{(n,q)}}. \tag{A.42}$$

$W^{1,(p,1)}$ *Bound for dw:* Applying the product rule yields

$$d(P^{-1}\Omega P) = dP^{-1}\Omega P + P^{-1}(d\Omega)P + P^{-1}\Omega dP. \tag{A.43}$$

Using $d\Omega \in L^{(p,1)}$ together with $W^{1,(p,1)} \hookrightarrow L^{(\frac{np}{n-p},q')}$:

$$\begin{aligned} \|d(P^{-1}\Omega P)\|_{L^{(p,1)}} &\leq C\Big(\|d\Omega\|_{L^{(p,1)}} + \|dP\|_{L^{(n,q)}} \|\Omega\|_{L^{(\frac{np}{n-p},q')}}\Big) \\ &\leq C\|\Omega\|_{W^{1,(p,1)}}\Big(1 + \|dP\|_{L^{(n,q)}}\Big). \end{aligned} \tag{A.44}$$

Again, by standard elliptic estimates we obtain:

$$\|dw\|_{W^{1,(p,1)}} \leq C\|\Delta w\|_{L^{(p,1)}} \leq C\|\Omega\|_{W^{1,(p,1)}}\Big(1 + \|P\|_{W^{1,(p,1)}}\Big). \tag{A.45}$$

Conclusion for ξ:

Note that by (A.4), we have the expression

$$*d\xi = dP^\top P + P^\top \Omega P \qquad \Leftrightarrow \qquad dP^\top = *d\xi P^\top - P^\top \Omega. \tag{A.46}$$

This leads to

$$\begin{aligned} \|dP\|_{L^{(n,q)}} &\lesssim \|d\xi\|_{L^{(n,q)}} + \|\Omega\|_{L^{(n,q)}}. \\ \|dP\|_{W^{1,(p,1)}} &\lesssim \|d\xi\|_{W^{1,(p,1)}} + \|\Omega\|_{W^{1,(p,1)}}. \end{aligned} \tag{A.47}$$

So once we established the bounds (A.5) and (A.6) for $d\xi$, we can conclude the proof by using (A.49).

By combining (A.40) and (A.42), we have

$$
\begin{aligned}
\|d\xi\|_{L^{(n,q)}} &\leq \|dv\|_{L^{(n,q)}} + \|dw\|_{L^{(n,q)}} \lesssim \delta\|dP\|_{L^{(n,q)}} + \|\Omega\|_{L^{(n,q)}} \\
&\overset{(A.47)}{\leq} C_1\delta\|d\xi\|_{L^{(n,q)}} + C_2(1+\delta)\|\Omega\|_{L^{(n,q)}}.
\end{aligned} \tag{A.48}
$$

Choosing δ sufficiently small such that $C_1\delta < \frac{1}{2}$, absorbing the term containing $d\xi$ to the LHS, we get (A.6).
In order to show (A.5), combining (A.41) and (A.45) yields

$$
\begin{aligned}
\|d\xi\|_{W^{1,(p,1)}} &\leq \|dv\|_{W^{1,(p,1)}} + \|dw\|_{W^{1,(p,1)}} \lesssim \delta\|dP\|_{W^{1,(p,1)}} + \|\Omega\|_{W^{1,(p,1)}} \\
&\overset{(A.47)}{\leq} C_3\delta\|d\xi\|_{W^{1,(p,1)}} + C_4\|\Omega\|_{W^{1,(p,1)}}.
\end{aligned} \tag{A.49}
$$

Again, choose δ small enough, such that also $C_3\delta < \frac{1}{2}$ holds, we can conclude (A.5).

□

Lemma A.3. *Let $1 \leq p < +\infty$, and let $B^n \subset \mathbb{R}^n$. Then, for any function $f \in L^p(B^n)$ which is a solution of*

$$-\Delta f = 0 \quad \text{on } B^n,$$

the function

$$r \mapsto \frac{1}{r^n}\int_{B_r} |f|^p \, dx^n,$$

is increasing.

Proof. We follow the proof in [9, Lemma 3.3.12].
Setting $v = |f|^p$, we first compute the Laplacian of v. Since f is harmonic ($\Delta f = 0$), a direct calculation gives

$$\operatorname{div}(\nabla v) = \operatorname{div}\left(p|f|^{p-2} f \nabla f\right) = \underbrace{p(p-1)|f|^{p-2}}_{\geq 0} \cdot \underbrace{|\nabla f|^2}_{\geq 0} + p|f|^{p-2} f \underbrace{\Delta f}_{=0} \geq 0. \tag{A.50}$$

Therefore, $\Delta v \geq 0$, so $v = |f|^p$ is subharmonic.
Integrating this inequality over the ball $B_r \subset \mathbb{R}^n$ and applying Stokes' theorem, we obtain

$$0 \leq \int_{B_r} \Delta v \, dx = \int_{\partial B_r} \frac{\partial v}{\partial r}(z)\, d\mu(z) = r^{n-1}\frac{d}{dr}\int_{\partial B^n} v(r\theta)\, d\mu(\theta), \tag{A.51}$$

where we parametrized the boundary by $z = r\theta$ for $\theta \in \partial B^n$ and used the fact that the surface measure scales as $d\mu(z) = r^{n-1} d\mu(\theta)$.
Equation (A.51) implies that

$$r \mapsto \int_{\partial B^n} |f(r\theta)|^p \, d\mu(\theta) \tag{A.52}$$

is non-decreasing. It follows that

$$\int_{B_r} |f(z)|^p\,dz = \int_0^r \rho^{n-1}\,d\rho \int_{\partial B^n} |f(\rho\theta)|^p\,d\mu(\theta) \overset{(A.52)}{\leq} \int_0^r \rho^{n-1}\,d\rho \int_{\partial B^n} |f(r\theta)|^p\,d\mu(\theta) \tag{A.53}$$

$$= \frac{r^n}{n}\int_{\partial B^n} |f(r\theta)|^p\,d\mu(\theta) = \frac{r^n}{n}\frac{1}{r^{n-1}}\int_{\partial B_r} |f(x)|^p\,d\mu(x) \tag{A.54}$$

$$= \frac{r}{n}\frac{d}{dr}\int_{B_r} |f(x)|^p\,dx. \tag{A.55}$$

Defining $F(r) := \int_{B_r} |f(x)|^p\,dx$, equation (A.55) can be rewritten as

$$F(r) \leq \frac{r}{n}F'(r) \quad \Leftrightarrow \quad 0 \leq \frac{F'(r)}{r^n} - \frac{nF(r)}{r^{n+1}} = \frac{d}{dr}\Big(\frac{F(r)}{r^n}\Big). \tag{A.56}$$

Integrating this differential inequality, we conclude that

$$r \mapsto \frac{1}{r^n}\int_{B_r} |f(x)|^p\,dx$$

is non-decreasing. □

Acknowledgements

The author would like to thank Francesca Da Lio and Tristan Rivière for their enriching discussions, ideas and feedback on previous versions of the paper and continued support.

The author would like to thank Dorian Martino for his advice and discussion on the AI generated Section 6.

On the Use of AI: All Sections, except Section 6 do not contain any proofs, strategies or ideas generated by AI. Upon completion of the paper, the author used ChatGPT-5.6 Sol Pro to show the optimality in Section 6. This Section was only verified by the author. The author used Gemini to find literature and typos.

References

[1] David R. Adams. "A note on Riesz potentials". In: *Duke Math. J.* 42.4 (1975), pp. 765–778. ISSN: 0012-7094,1547-7398. URL: http://projecteuclid.org/euclid.dmj/1077311348.

[2] Colin Bennett and Robert Sharpley. *Interpolation of operators*. Vol. 129. Pure and Applied Mathematics. Academic Press, Inc., Boston, MA, 1988, pp. xiv+469. ISBN: 0-12-088730-4.

[3] Yann Bernard et al. *The Regularity of Critical Points to Scale-Invariant Curvature Energies in Dimension 4*. 2026. arXiv: 2511.01765 [math.AP]. URL: https://arxiv.org/abs/2511.01765.

[4] Fabrice Bethuel. "Un résultat de régularité pour les solutions de l'équation de surfaces à courbure moyenne prescrite". In: *C. R. Acad. Sci. Paris Sér. I Math.* 314.13 (1992), pp. 1003–1007. ISSN: 0764-4442.

[5] R. Coifman et al. "Compensated compactness and Hardy spaces". In: *J. Math. Pures Appl. (9)* 72.3 (1993), pp. 247–286. ISSN: 0021-7824.

[6] Loukas Grafakos. *Classical Fourier analysis*. Second. Vol. 249. Graduate Texts in Mathematics. Springer, New York, 2008, pp. xvi+489. ISBN: 978-0-387-09431-1.

[7] Chang-Yu Guo and Chang-Lin Xiang. "Conservation law of harmonic mappings in supercritical dimensions". In: *C. R. Math. Acad. Sci. Paris* 362 (2024), pp. 769–773. ISSN: 1631-073X,1778-3569. DOI: 10.5802/crmath.592. URL: https://doi.org/10.5802/crmath.592.

[8] Frédéric Hélein. "Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne". In: *C. R. Acad. Sci. Paris Sér. I Math.* 312.8 (1991), pp. 591–596. ISSN: 0764-4442.

[9] Frédéric Hélein. *Harmonic maps, conservation laws and moving frames*. Second. Vol. 150. Cambridge Tracts in Mathematics. Translated from the 1996 French original, With a foreword by James Eells. Cambridge University Press, Cambridge, 2002, pp. xxvi+264. ISBN: 0-521-81160-0. DOI: 10.1017/CBO9780511543036. URL: https://doi.org/10.1017/CBO9780511543036.

[10] T. Iwaniec, C. Scott, and B. Stroffolini. "Nonlinear Hodge theory on manifolds with boundary". In: *Ann. Mat. Pura Appl. (4)* 177 (1999), pp. 37–115. ISSN: 0003-4622. DOI: 10.1007/BF02505905. URL: https://doi.org/10.1007/BF02505905.

[11] Tadeusz Iwaniec and Gaven Martin. *Geometric function theory and non-linear analysis*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001, pp. xvi+552. ISBN: 0-19-850929-4.

[12] Dorian Martino and Armin Schikorra. "Regularizing properties of n-Laplace systems with antisymmetric potentials in Lorentz spaces". In: *Math. Ann.* 389.4 (2024), pp. 3301–3348. ISSN: 0025-5831,1432-1807. DOI: 10.1007/s00208-023-02727-2. URL: https://doi.org/10.1007/s00208-023-02727-2.

[13] Dorian Martino and Armin Schikorra. *Non-Regularizing properties of n-Laplace systems with antisymmetric potentials in critical Lebesgue spaces*. 2026. arXiv: 2608.23105 [math.AP]. URL: https://arxiv.org/abs/2608.23105.

[14] Charles B. Morrey Jr. *Multiple integrals in the calculus of variations*. Classics in Mathematics. Reprint of the 1966 edition [MR0202511]. Springer-Verlag, Berlin, 2008, pp. x+506. ISBN: 978-3-540-69915-6. DOI: 10.1007/978-3-540-69952-1. URL: https://doi.org/10.1007/978-3-540-69952-1.

[15] Richard O'Neil. "Convolution operators and $L(p, q)$ spaces". In: *Duke Math. J.* 30 (1963), pp. 129–142. ISSN: 0012-7094,1547-7398. URL: http://projecteuclid.org/euclid.dmj/1077374532.

[16] Jaak Peetre. "Espaces d'interpolation et théorème de Soboleff". In: *Ann. Inst. Fourier (Grenoble)* 16 (1966), pp. 279–317. ISSN: 0373-0956,1777-5310. URL: http://www.numdam.org/item?id=AIF_1966__16_1_279_0.

[17] S. Poornima. "An embedding theorem for the Sobolev space $W^{1,1}$". In: *Bull. Sci. Math. (2)* 107.3 (1983), pp. 253–259. ISSN: 0007-4497.

[18] Tristan Rivière. "Conservation laws for conformally invariant variational problems". In: *Invent. Math.* 168.1 (2007), pp. 1–22. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-006-0023-0. URL: https://doi.org/10.1007/s00222-006-0023-0.

[19] Tristan Rivière. "The role of conservation laws in the analysis of conformally invariant problems". In: *Topics in modern regularity theory.* Vol. 13. CRM Series. Ed. Norm., Pisa, 2012, pp. 117–167. ISBN: 978-88-7642-426-7; 88-7642-426-7. DOI: 10.1007/978-88-7642-427-4_2. URL: https://doi.org/10.1007/978-88-7642-427-4_2.

[20] Tristan Rivière and Michael Struwe. "Partial regularity for harmonic maps and related problems". In: *Comm. Pure Appl. Math.* 61.4 (2008), pp. 451–463. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.20205. URL: https://doi.org/10.1002/cpa.20205.

[21] Armin Schikorra. "A remark on gauge transformations and the moving frame method". In: *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 27.2 (2010), pp. 503–515. ISSN: 0294-1449,1873-1430. DOI: 10.1016/j.anihpc.2009.09.004. URL: https://doi.org/10.1016/j.anihpc.2009.09.004.

[22] Dominik Schlagenhauf. *A Discontinuous Solution of the Critical n-Laplace System with Antisymmetric Potential.* 2026. arXiv: 2608.24393 [math.AP]. URL: https://arxiv.org/abs/2608.24393.

[23] E. M. Stein. "Editor's note: the differentiability of functions in $\mathbf{R}^n$". In: *Ann. of Math. (2)* 113.2 (1981), pp. 383–385. ISSN: 0003-486X. URL: http://links.jstor.org/sici?sici=0003-486X(198103)2:113:2%3C383:ENTDOF%3E2.0.CO;2-4&origin=MSN.

[24] Karen K. Uhlenbeck. "Connections with L^p bounds on curvature". In: *Comm. Math. Phys.* 83.1 (1982), pp. 31–42. ISSN: 0010-3616,1432-0916. URL: http://projecteuclid.org/euclid.cmp/1103920743.

[25] Henry C. Wente. "An existence theorem for surfaces of constant mean curvature". In: *J. Math. Anal. Appl.* 26 (1969), pp. 318–344. ISSN: 0022-247X. DOI: 10.1016/0022-247X(69)90156-5. URL: https://doi.org/10.1016/0022-247X(69)90156-5.

DEPARTMENT OF MATHEMATICS, ETH ZENTRUM, CH-8093 ZÜRICH, SWITZERLAND
Email address: carolin.bayer@math.ethz.ch